Computational Higher-Dimensional Algebra packages at Bangor

The CHDA packages developed at Bangor over the last ten years are:

- XMod [2] – crossed modules and cat^1-groups, with Murat Alp [1, 3],
- Gpd [13] – groupoids, group graphs and groupoid graphs, with Emma Moore [12, 6],
- Kan [9] – double coset rewriting systems for fp-groups, with Anne Heyworth [8, 5],
- IdRel [10] – logged rewriting and identities among relators with Anne Heyworth [8, 11].

The work with these three students at Bangor has been based on research in Algebraic Topology and Category Theory developed by Ronnie Brown, Tim Porter, their graduate students and postdocs.
Another thesis of particular relevance is that of Magnus Forrester-Barker [7] on representations of crossed modules.

This talk describes recent and planned developments for XMod and Gpd, including an implementation of crossed modules over groupoids.
In particular, we present results from [4] on automorphisms of these structures, written while Murat Alp was visiting Bangor in July this year.

Extensive background material is provided in the working document [14].

Review of crossed modules and cat[1]-groupoids

A crossed module has the form:

$$\mathcal{X} \quad = \quad \begin{array}{c} S \\ \Big\downarrow \partial \\ R \end{array} \qquad \text{where} \quad \partial(s^r) = (\partial s)^r \quad \text{and} \quad s^{(\partial s')} = s^{s'}.$$

Note that an action of R on S is required,
while S, R both act on themselves by conjugation.

The equivalent cat[1]-group is given by:

$$\mathcal{C} \quad = \quad \begin{array}{c} G \cong R \ltimes S \\ e\Big\uparrow \ \Big\downarrow\Big\downarrow t,h \\ R \end{array} \qquad \text{where} \quad \begin{array}{l} e * t = \mathrm{id}_R, \\ e * h \in \mathrm{Aut} R, \\ [\ker t, \ker h] = 1. \end{array}$$

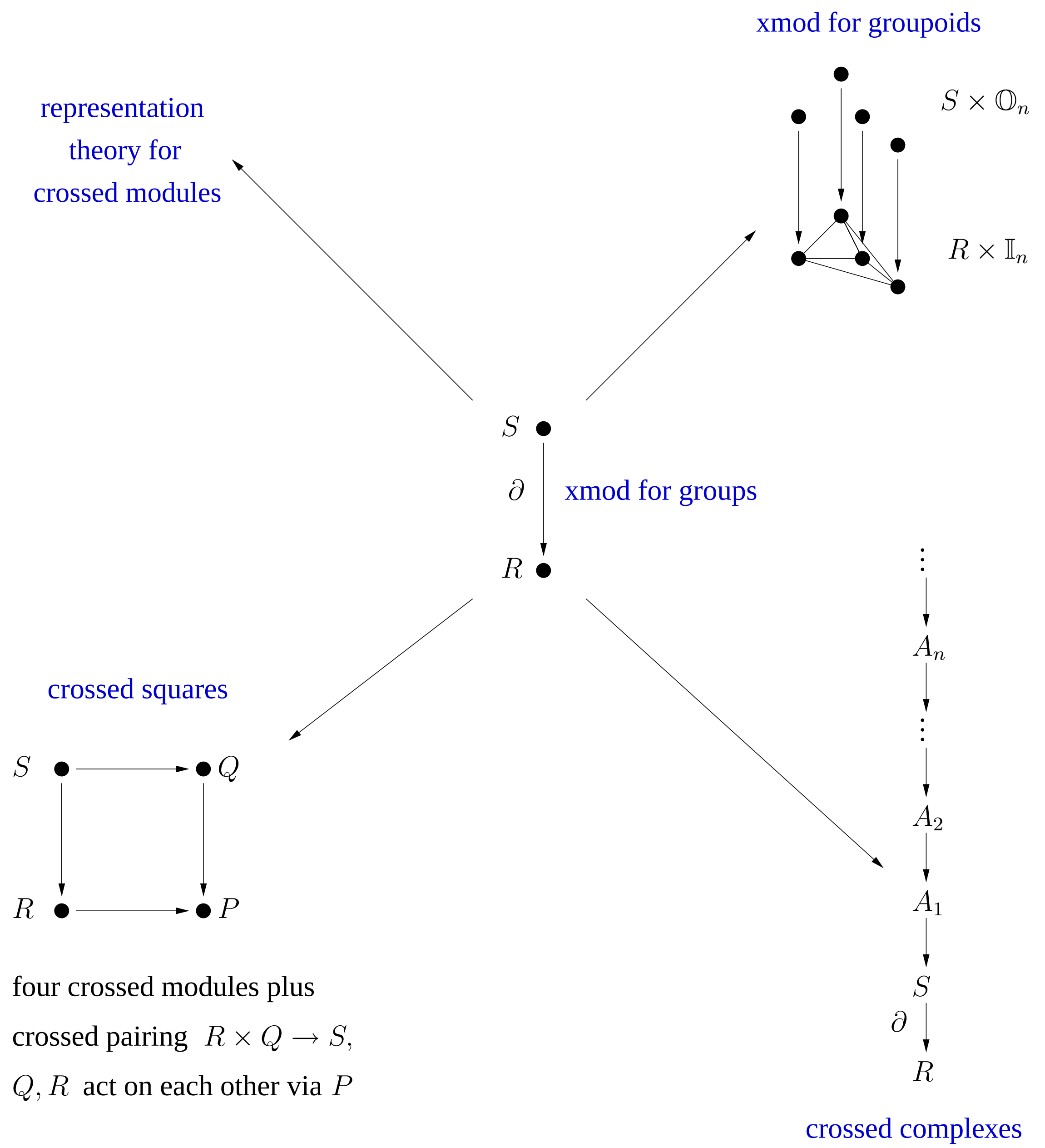
Work in this area typically requires conversion from one category to an equivalent one.
Here, we construct $\mathcal{C}$ from $\mathcal{X}$ by $G = R \ltimes S$ and

$$t(r,s) = r, \qquad h(r,s) = r(\partial s), \qquad er = (r,1),$$

and $\mathcal{X}$ from $\mathcal{C}$ by:

$$S = \ker t, \qquad \partial = h|_S, \qquad s^r = (er)^{-1} s(er).$$

Planned generalisations and extensions



Some code for crossed squares is already in the development version of XMod.
Today we will restrict attention to crossed modules over groupoids, and their automorphisms.

The Bangor situation

The sorry state of affairs at Bangor is that the sequence of theses described earlier has come to a halt. The Mathematics Department has closed, the last mathematics undergraduates finished their degrees this summer, and the remaining staff have been persuaded that they were only too happy to accept the offered early retirement packages!

The result for me is that I can continue to teach the occasional discrete mathematics module for first-year computer science students and devote the rest of my time to research and GAP programming. Progress is therefore at last possible with my list of "things to do" with the CHDA packages.

Thus this workshop comes at an opportune time, and I am most grateful to the organisers for what has been a splendid week in Braunschweig.

Basic notation for groupoids

A *groupoid* is a category in which every arrow is invertible. In the notation used here, a groupoid $\mathbb{C} = (C_1, C_0)$ consists of the following:

- a set $\mathrm{Ob}(\mathbb{C}) = C_0$ of *objects*,
- a set $\mathrm{Arr}(\mathbb{C}) = C_1$ of *arrows*,
- *source* and *target* maps $s, t : C_1 \to C_0$, so that we write $(a : u \to v)$ whenever $sa = u$ and $ta = v$, and we denote by $\mathbb{C}(u, v)$ the *hom-set* of arrows with source u and target v,
- an *identity arrow* 1_u at each object u, with $s1_u = t1_u = u$,
- an associative partial composition $C_1 \times_0 C_1 \to C_1$, with ab defined whenever $ta = sb$, such that $s(ab) = sa$ and $t(ab) = tb$, so that $\mathbb{C}(u) := \mathbb{C}(u, u)$ is a group, called the *object group* at u,
- for each arrow $(a : u \to v)$ an inverse arrow $(a^{-1} : v \to u)$ such that $aa^{-1} = 1_u$ and $a^{-1}a = 1_v$.

A morphism of groupoids, as for general categories, is called a functor.
Thus a *functor* $\phi = (\phi_1, \phi_0) : \mathbb{C} \to \mathbb{D}$ is a pair of maps $(\phi_1 : C_1 \to D_1,\ \phi_0 : C_0 \to D_0)$ such that $\phi_1 1_u = 1_{\phi_0 u}$ and $\phi_1(ab) = (\phi_1 a)(\phi_1 b)$ whenever the composite arrow is defined.
It is often convenient to omit the subscripts $0, 1$ since it should be clear from the context whether an object or an arrow is being mapped.
ϕ is *injective* and/or *surjective* if both ϕ_0, ϕ_1 are.
Automorphisms of $\mathbb{C}$ are bijective functors $\mathbb{C} \to \mathbb{C}$.

Example 1

The categories of groups and groupoids, and their morphisms, are written $\mathbf{Gp}$, $\mathbf{Gpd}$ respectively.
There is a functor Gpd: $\mathbf{Gp} \to \mathbf{Gpd}$, $G \mapsto \mathbb{G}$, where $\mathbb{G}$ is a groupoid with a single object (written '$*$' or '$\bullet$').

Example 2

For X a set, the *trivial groupoid* $\mathbb{O}(X)$ *on* X has $\mathrm{Ob}(\mathbb{O}) = X$ and $\mathrm{Arr}(\mathbb{O}) = \{1_x \mid x \in X\}$.
We denote $\mathbb{O}(\{1, \ldots, n\})$ by $\mathbb{O}_n$.

Example 3

The *unit groupoid* $\mathbb{I}$ has objects $\{0, 1\}$ and four arrows. The two non-identity arrows are $(\iota : 0 \to 1)$ and its inverse $(\iota^{-1} : 1 \to 0)$.

The *underlying digraph* of a groupoid is obtained by forgetting the composition, so the objects become vertices, the arrows become arcs, while the source and target maps have their usual digraph meaning. A groupoid is *connected* if its underlying digraph is.

Example 4

The *tree groupoid* $\mathbb{I}_n$ has n objects $\{1, 2, \ldots, n\}$ and n^2 arrows $\{(p, q) \mid 1 \leqslant p, q \leqslant n\}$ where $s(p, q) = p$, $t(p, q) = q$, $(p, q)(q, r) = (p, r)$, and $(p, q)^{-1} = (q, p)$. Note that $\mathbb{I}_2 \cong \mathbb{I}$. We also write $\mathbb{I}(X)$ for the tree groupoid on a set of objects X. The underlying digraph of $\mathbb{I}_n$ is complete.

The *product* $\mathbb{C} \times \mathbb{D}$ of groupoids $\mathbb{C}, \mathbb{D}$ has objects $C_0 \times D_0$, arrows $C_1 \times D_1$, and composition $(a_1, b_1)(a_2, b_2) = (a_1a_2, b_1b_2)$, so $(a, b)^{-1} = (a^{-1}, b^{-1})$.

Example 5

If $\mathbb{G}$ is a group, considered as a one-object groupoid, and $\mathbb{I}_n$ is a tree groupoid, then $\mathbb{C} = \mathbb{G} \times \mathbb{I}_n$ may be thought of as the groupoid with n objects $\{1, 2, \ldots, n\}$ and $n^2|G|$ arrows $\{(i, g, j) \mid g \in G, 1 \leqslant i, j \leqslant n\}$, with

source: $s(i, g, j) = i$, target: $t(i, g, j) = j$,

composition: $(i, g, j)(j, h, k) = (i, gh, k)$,

and inverses: $(i, g, j)^{-1} = (j, g^{-1}, i)$.

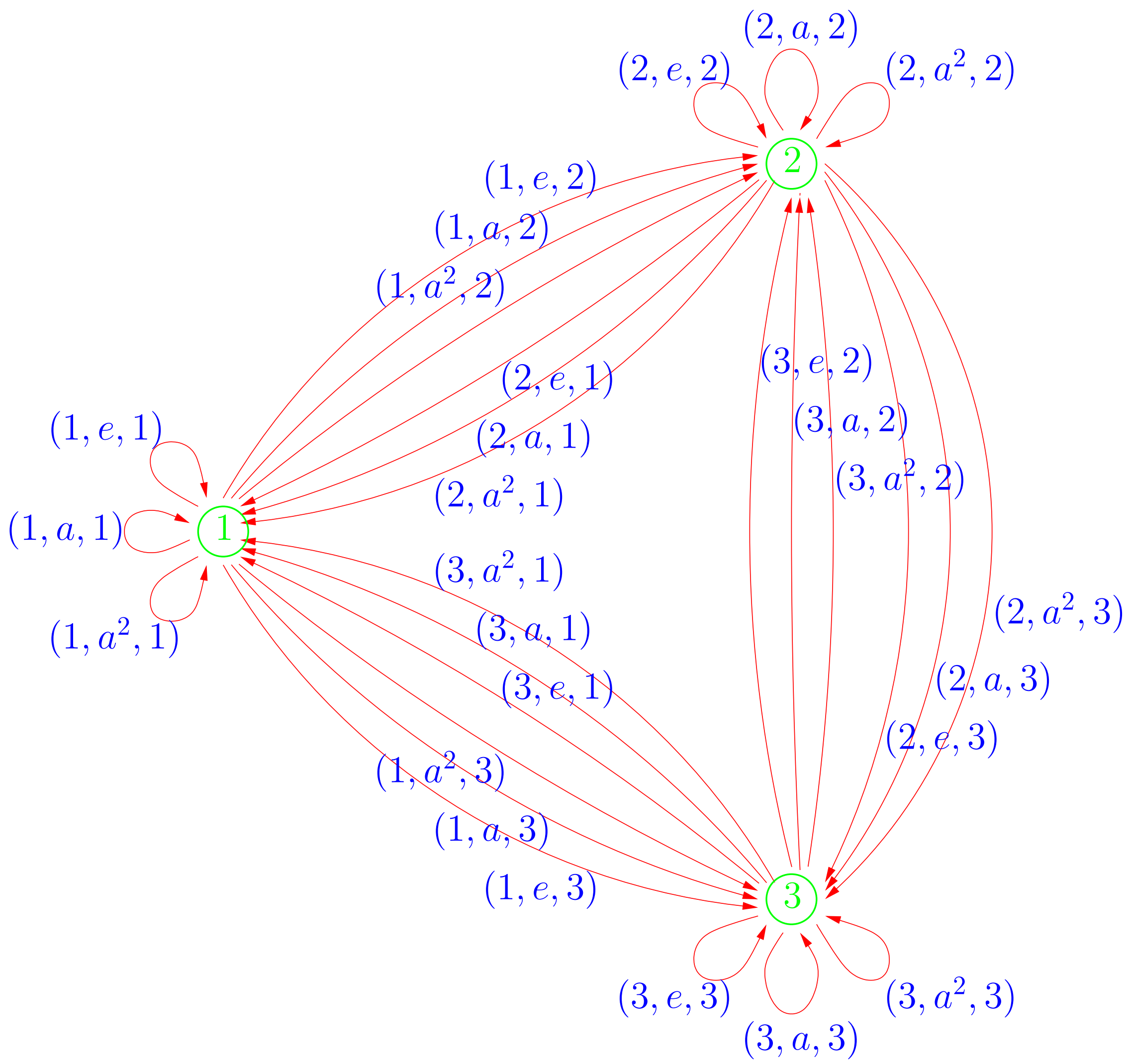
A generating set for $\mathbb{G}$ is given by $\{(1, g, 1) \mid g \in X_G\} \cup X_n$ where X_G is any generating set for $\mathbb{G}$ and $X_n = \{(1, e, j) \mid 2 \leqslant j \leqslant n\}$.

Every finite, connected groupoid is isomorphic to a direct product of a group and a tree groupoid in this way, and we call such a representation a *standard connected groupoid*.

In general a groupoid is a disjoint union of connected groupoids, but the only non-connected groupoids considered today are the $\mathbb{O}_n$.

Example 6: a standard connected groupoid

A very simple example is given by the groupoid $C_3 \times \mathbb{I}_3$, where $C_3 = \{e, a, a^2\}$, for which we may sketch the following diagram:



Composition is: $(i,g,j)(j,h,k) = (i,gh,k)$.

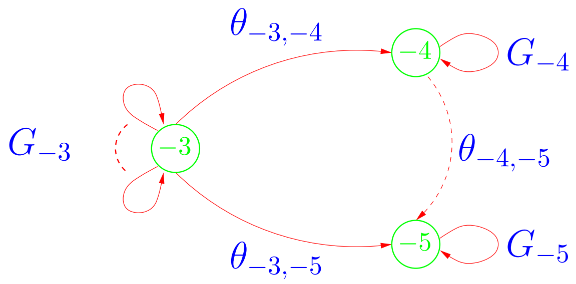
The number of arrows is: $|G|.(\#\text{ objects})^2 = 27$.

Implementation of a connected groupoid requires:

- a set O of objects (some negative integers, say),
- a group G_i at a chosen root object i,
- a tree of isomorphisms θ_{ij} $(i \neq j)$.

Current version:

- standard connected groupoid $\mathbb{G} \times \mathbb{I}_n$,
- the θ_{ij} are all identity maps,



$$\theta_{j,k} = \theta_{i,j}^{-1} * \theta_{ik}$$

Old version (to be re-introduced soon):

- $G_i, G_j, \ldots$ conjugate in some parent group P,
- θ_{ij} is conjugation by $p_{ij} \in P$.

One day, more generally:

- $G_i, G_j, \ldots$ any isomorphic groups,
- $\mathrm{Hom}(i,j) = \{(i,g,j) \mid i,j \in O,\ g \in G_j\}$,
- composition: $(i,g,j)(j,h,k) = (i,(\theta_{jk}g)h,k)$.

Automorphism groupoid of a group

Groupoids are a "good thing" because the category of groupoids is cartesian closed. This means, in particular, that the "automorphism gadget" of a group G may be thought of as a groupoid where

- the objects are the automorphisms of G,
- the arrows are natural isomorphisms.

We recall the basic ideas of natural isomorphisms. If $\beta, \gamma : \mathbb{C} \to \mathbb{D}$ are functors, then a natural transformation $\tau : \beta \to \gamma$ is determined by a function $\tau : \mathrm{Ob}(\mathbb{C}) \to \mathrm{Arr}(\mathbb{D}),\ u \mapsto \tau_u$, such that the following diagram commutes for every $(a : u \to v) \in \mathbb{C}$,

$$\begin{tikzcd}
\beta u \arrow[r, "\tau_u"] \arrow[d, "\beta a"'] \arrow[dr, "\tau a"] & \gamma u \arrow[d, "\gamma a"] \\
\beta v \arrow[r, "\tau_v"'] & \gamma v
\end{tikzcd}$$

List $[s\tau_{u_1}, \ldots, s\tau_{u_n}]$ is a permutation of $\{u_1, \ldots, u_n\}$, the objects of $\mathbb{C}$. Commutativity of the diagram allows us to extend τ to a function $\mathrm{Arr}(\mathbb{C}) \to \mathrm{Arr}(\mathbb{D})$, $a \mapsto \tau a$, where $\tau 1_u = \tau_u$ for each object u.

Restricting to groupoids (so arrows are invertible) we have $\tau_v = (\beta a)^{-1}(\tau_u)(\gamma a)$. So τ is fixed if we are given, for each component of $\mathbb{C}$, the image of one object. Furthermore, when β, γ are surjective, every transformation is invertible with $(\tau^{-1})_u = (\tau_u)^{-1}$,

$$\begin{array}{ccc} \gamma u & \xrightarrow{(\tau^{-1})_u = \tau_u^{-1}} & \beta u \\ {\scriptstyle \gamma a}\downarrow & & \downarrow{\scriptstyle \beta a} \\ \gamma v & \xrightarrow[(\tau^{-1})_v = \tau_v^{-1}]{} & \beta v \end{array}$$

and we call τ a *natural equivalence*.
Then $[t\tau_{u_1}, t\tau_{u_2}, \ldots, t\tau_{u_n}]$ is a permutation of $\mathrm{Ob}(\mathbb{D})$.

Natural equivalences compose in the obvious way. If δ is a third functor from $\mathbb{C}$ to $\mathbb{D}$, and if $\sigma : \gamma \to \delta$ is a second natural equivalence, then we obtain the diagrams:

$$\begin{tikzcd} \beta u \arrow[r, "\tau_u"] \arrow[d, "\beta a"'] \arrow[dr, "\tau a"] & \gamma u \arrow[r, "\sigma_u"] \arrow[d, "\gamma a"'] \arrow[dr, "\sigma a"] & \delta u \arrow[d, "\delta a"] \\ \beta v \arrow[r, "\tau_v"'] & \gamma v \arrow[r, "\sigma_v"'] & \delta v \end{tikzcd} \qquad \begin{tikzcd} C_1 \arrow[r, "{\beta_1, \gamma_1, \delta_1}"] \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] & C_1 \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] \\ C_0 \arrow[r, "{\beta_0, \gamma_0, \delta_0}"'] \arrow[ur, "{\tau, \sigma}"] & C_0 \end{tikzcd}$$

Thus we obtain a composite natural equivalence $\tau\sigma \ : \ \beta \to \delta$ where:

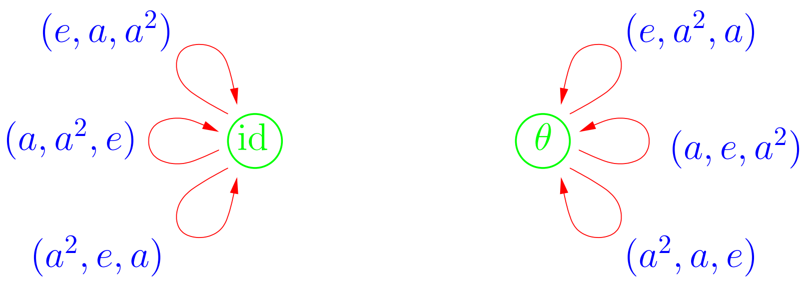
$$\begin{aligned}(\tau\sigma)u &= \tau_u\sigma_u, \\ (\tau\sigma)a &= (\tau a)(\sigma_v) \ = \ (\tau_u)(\sigma a) \ = \ (\tau a)(\gamma a)^{-1}(\sigma a).\end{aligned}$$

We thus obtain a groupoid whose objects are isomorphisms and whose arrows are natural equivalences. When $\mathbb{C} = \mathbb{D}$ and we obtain the *automorphism groupoid* $\mathrm{AUT}\,\mathbb{C}$ of $\mathbb{C}$.

Example 7

The group $C_3 = \{e, a, a^2\}$ has two automorphisms: the identity id and $\theta : a \mapsto a^2$, so $\mathrm{Aut}(C_3) \cong C_2$.

We represent $\tau : C_3 \to C_3$ by the triple $(\tau e, \tau a, \tau a^2)$. There are six such τ, as shown below:



This is the group-groupoid $\mathrm{AUT}\,C_3$ associated to the crossed module $(0 : C_3 \to C_2)$. The corresponding cat^1-group has source $C_2 \ltimes C_3 \cong S_3$.

Automorphisms of $\mathbb{C} = \mathbb{G} \times \mathbb{I}_n$.

An automorphism $\alpha \in A = \mathrm{Aut}(\mathbb{C})$ is required to preserve the groupoid structure:

$$s(\alpha a) = \alpha(sa), \;\; t(\alpha a) = \alpha(ta), \;\; \alpha(ab) = (\alpha a)(\alpha b).$$

There are three types of automorphism of $\mathbb{C}$:

(1) For π a permutation in the symmetric group S_n we define the automorphism α_π by

$$\alpha_\pi(i, g, j) \;=\; (\pi i, g, \pi j).$$

(2) We may apply an automorphism κ of G to the loops at root object 1, giving an automorphism α_κ of $\mathbb{C}$, fixing the objects, where

$$\alpha_\kappa(1, g, 1) \;=\; (1, \kappa g, 1), \quad \alpha_\kappa(1, e, j) \;=\; (1, e, j).$$

It follows that $\alpha_\kappa(i, g, j) = (i, \kappa g, j)$, so α_κ applies κ to all the hom-sets at once.

(3) Hom-set $\mathbb{C}(i, j)$ gives a regular representation for G with action $(i, g, j)^h = (i, gh, j)$. Automorphisms $\alpha_{\boldsymbol{b}}$ are defined for each $\boldsymbol{b} = (b_1, \ldots, b_n) \in G^n$ by

$$\alpha_{\boldsymbol{b}}(i, g, j) \;=\; (i, b_i^{-1} g b_j, j).$$

Theorem 1 $\mathrm{Aut}\,\mathbb{C} \;\cong\; (S_n \times \mathrm{Out}\,G) \ltimes Q$.

Here is an outline of this construction.

There are actions of both S_n and $\mathrm{Aut}\,G$ on G^n, where

$$\begin{aligned} \boldsymbol{b}^{\pi} &= (b_{\pi^{-1}1}, \ldots, b_{\pi^{-1}i}, \ldots, b_{\pi^{-1}n}), \\ \boldsymbol{b}^{\kappa} &= (\kappa b_1, \ldots, \kappa b_i, \ldots, \kappa b_n), \end{aligned}$$

and these actions commute, giving an action of $S_n \times \mathrm{Aut}\,G$ on G^n.

The map $\theta : G^n \to \mathrm{Aut}\,\mathbb{C},\ \boldsymbol{b} \mapsto \alpha_{\boldsymbol{b}}$ is a homomorphism, and $\ker\theta$ is the set of constant vector $(z, z, \ldots, z)$ with $z \in Z(G)$, the centre of G.

When $\boldsymbol{g} = (g, g, \ldots, g)$ is an arbitrary constant vector in G^n, the type (3) $\alpha_{\boldsymbol{g}}$ is also the type (2) conjugation automorphism $\alpha_{(\wedge g)}$. We denote by $\hat{G}$ the diagonal subgroup in G^n, put $\hat{Z} = \ker\theta$, and define $Q = G^n/\hat{Z}$.

An automorphism $\alpha = (\alpha_1, \alpha_0)$ of $\mathbb{C}$ is specified by:

- $\kappa \in \mathrm{Aut}(G)$, so that $\alpha_1(1, g, 1) = (\pi 1, \kappa g, \pi 1)$,
- images $\alpha_1(1, e, j) = (\pi 1, b_j, \pi j),\ 2 \leqslant j \leqslant n$,
- the permutation $\alpha_0 = \pi$ on the objects,

and so α has the form $\alpha_\kappa * \alpha_{\boldsymbol{b}} * \alpha_\pi$ with $b_1 = e$.

Natural equivalences for $\mathbb{C}$

If automorphisms β, γ of $\mathbb{C}$ have the form

$$\beta = \alpha_\kappa * \alpha_{\boldsymbol{b}} * \alpha_\pi, \qquad \gamma = \alpha_\lambda * \alpha_{\boldsymbol{c}} * \alpha_\xi$$

then there is a natural equivalence $\tau : \beta \to \gamma$ with $\tau_1 = (\pi 1, h, \xi 1)$ provided

$$\lambda g = (\kappa g)^h, \qquad \tau_j = (\pi j, b_j^{-1} h c_j, \xi j).$$

It follows that $\mathrm{AUT}\,\mathbb{C}$ has $|\mathrm{Out} G|$ components; vertex groups $Z = Z(G)$; with $|S_n \ltimes Q|$ objects in each component.

Applying this to $\mathbb{C} = C_3 \times \mathbb{I}_3$ of Example 6, we find

$$A \;=\; \mathrm{Aut}(C_3 \times \mathbb{I}_3) \;\cong\; (S_3 \times C_2) \ltimes C_3^2,$$

so $|A| = 108$.

These 108 automorphisms are the objects in the automorphism groupoid, and form two components with 54 objects in each component, so that

$$\mathrm{AUT}(C_3 \times \mathbb{I}_3) \;\cong\; 2 \text{ copies of } C_3 \times \mathbb{I}_{54}.$$

Compare this with $\mathrm{AUT}\,C_3$ in Example 7.

Here is the general result.

Theorem 2

The automorphism groupoid $\mathrm{AUT}\,\mathbb{C}$ of $\mathbb{C} = \mathbb{G} \times \mathbb{I}_n$ has

- $n!.|\mathrm{Aut}\,G|.|G|^{n-1}$ objects (automorphisms),
- $(n!)^2.|\mathrm{Aut}\,G|.|G|^{2n-1}$ arrows (natural equiv.),
- degree $|Z(G)| = |G|/|\mathrm{Inn}\,G|$,
- $|\mathrm{Out}\,G|$ connected components, with $n!.|\mathrm{Inn}\,G|.|G|^{n-1}$ objects in each component.

Corollary

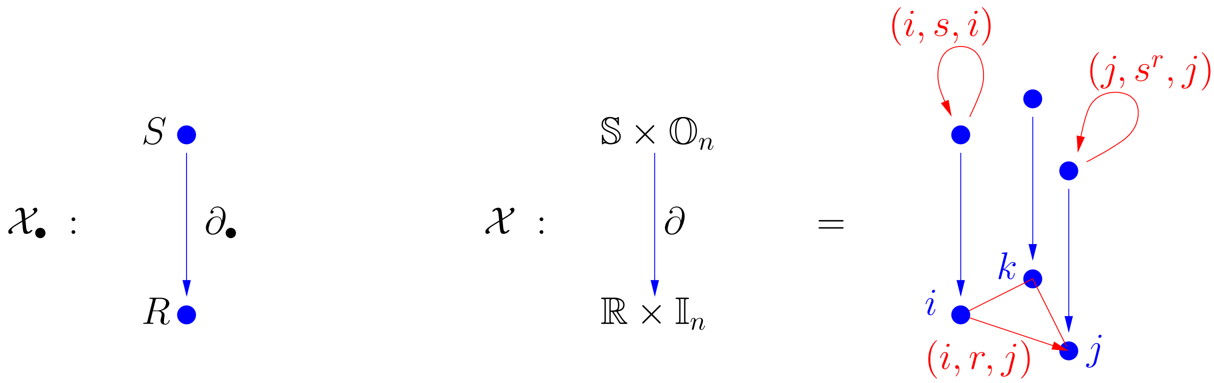
When $\mathbb{C}$ is a group G considered as a one-object groupoid, the automorphism groupoid has

- $|\mathrm{Aut}\,G|$ objects;
- $|\mathrm{Aut}\,G|.|G|$ natural equivalences;
- degree $|Z(G)|$;
- $|\mathrm{Out}\,G|$ components;
- $|\mathrm{Inn}\,G|$ objects in each component.

Automorphisms of a crossed module of groupoids

From $\mathcal{X}_\bullet = (\partial_\bullet : S \to R)$ we construct

$$\mathcal{X} \;=\; (\partial \;:\; (\mathbb{S} \times \mathbb{O}_n) \longrightarrow (\mathbb{R} \times \mathbb{I}_n)) \;.$$



The groupoid action is given by

$$(i, s, i)^{(i,r,j)} \;:=\; (j, s^r, j).$$

An automorphism of $\mathcal{X}$ is a triple $(\alpha_2, \alpha_1, \alpha_0)$ satisfying:

- α_0 is a permutation of the objects,
- $(\alpha_1, \alpha_0) \in \mathrm{Aut}(\mathbb{R} \times \mathbb{I}_n)$, a groupoid automorphism,
- $(\alpha_2, \alpha_0) \in \mathrm{Aut}(\mathbb{S} \times \mathbb{O}_n)$, a groupoid automorphism,
- $\alpha_2 * \partial \;=\; \partial * \alpha_1$,
- $\alpha_2 \left((i, s, i)^{(i,r,j)}\right) \;=\; (\alpha_2(i, s, i))^{\alpha_1(i,r,j)}$.

The automorphisms of $\mathcal{X}$ are known once those of $\mathcal{X}_\bullet,\ \mathbb{O}_n,\ \mathbb{I}_n$ are known.

Again there are three types of automorphism:

(1) $\pi \in S_n \ \Rightarrow\ (\alpha_\pi, \alpha_\pi, \pi) \in \mathrm{Aut}\boldsymbol{\mathcal{X}}$,

(2) $(\sigma, \rho) \in \mathrm{Aut}\boldsymbol{\mathcal{X}_\bullet} \ \Rightarrow\ (\alpha_\sigma, \alpha_\rho, (\ )) \in \mathrm{Aut}\boldsymbol{\mathcal{X}}$,

(3) $\boldsymbol{b} \in R^n \ \Rightarrow\ ((\alpha_{\boldsymbol{b}}, \alpha_{\boldsymbol{b}}, (\ )) \in \mathrm{Aut}\boldsymbol{\mathcal{X}}$
where $\alpha_{\boldsymbol{b}}(i, s, i) := (i, s^{b_i}, i)$.

The automorphisms $\{(\wedge r, \wedge r) \mid r \in R\}$ form a normal subgroup N of $\mathrm{Aut}\mathcal{X}_\bullet$.

The following result resembles Theorem 2.

Theorem 3

$$\mathrm{Aut}\,\boldsymbol{\mathcal{X}} \ \cong\ (S_n \times N) \ltimes (R^n/\hat{Z}),$$

where now Z is the centre of R.

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