On the Genus of a p-Group

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Introduction

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- G is a finite group acting on Σ_g faithfully and preserving the orientation.
- Then G can be realized as a group of orientation preserving conformal homeomorphisms $f : X \longrightarrow X$ for some Riemann Surface structure (X, Σ_g) on the underlying space Σ_g.

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- A Reverse Question :

  Given a finite group G, to find out the list of numbers $g \geq 2$ such that G acts on Σ_g.

Introduction : Hurwitz Theorem

A finite group G acts on a surface Σ_g with quotient $\Sigma_g/G \approx \Sigma_h$ if and only if

$$2(g-1) = |G|\{2(h-1)+\Sigma_{i=1}^{r}(1-\frac{1}{n_i})\} \quad \text{(R. H. Formula)}$$

and there exist elements $a_1, b_1, \cdots, a_h, b_h, x_1, \cdots, x_r$ in G such that

$G = \langle a_1, b_1, \cdots, a_h, b_h, x_1, \cdots, x_r \rangle$, and that

$$\prod_{i=1}^{h}(a_i, b_i)\prod_{j=1}^{r}x_j = 1 \qquad \text{(Long Relation)}$$

where x_i is of order n_i in G (Order Relation).

Data Spectrum of a Group G

- A data D associated to G is $(h; n_1, n_2, \ldots, n_r)$ s.t. :
  - G is generated by non-trivial elements :
    $$a_1, b_1, \ldots, a_h, b_h, x_1, \ldots, x_r; h, r \geq 0$$
  - Order of x_i is $n_i,\ 1 \leq i \leq r$, and
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    $$\prod_{i=1}^{h} [a_i, b_i] \prod_{j=1}^{r} x_j = 1$$
- The number g is called genus of G associated to D if

$$g(D) := g = |G|\{(h-1) + \frac{1}{2}\Sigma_{i=1}^{r}(1 - \frac{1}{n_i})\}$$

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- Kulkarni (1987) showed that there is an $N = N(G)$ such that :
  - $g \in \mathrm{sp(G)} \implies g \equiv 1 \bmod N$
  - for almost all $g \equiv 1 \bmod N$, one has $g \in \mathrm{sp(G)}$.

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- This specifies the following :
  - Minimum genus : $\mu_0(G)$ = minimum $\mathrm{sp(G)}$
  - Minimum stable genus : $\sigma_0(G)$

    = minimum

$\{l \geq 2 : l \in \mathrm{sp(G)},\ g \geq l, g \equiv 1 \bmod N \implies g \in \mathrm{sp(G)}\}$

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- Known Results :
  - Cyclic p-Groups Kulkarni, Maclachlan, 1991
  - Groups with MEP Maclachlan, Talu, 1998,

    Detailed calculations are done for :
    - Elementary Abelian p-Groups
    - p-Groups with proper cyclic subgroups of index p (and p^2)
  - Split Metacyclic Groups Weaver, 2001
  - p-groups of exponent p Oesterlé, S.
  - p-groups of maximal class of order upto p^p S.

p-Groups of exponent p

- The R. H. Formula in Hurwitz Theorem is of the form :

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- Associated Graded Lie Algebra (over $\mathbb{F}_p$)

$$A(G) = \bigoplus_{n\geq 1} A^n(G), \quad A^n(G) = \gamma_n(G)/\gamma_{n+1}(G)$$

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- Semi-rank of a finite p-group depends on $A^1(G), A^2(G)$ and the grading among them.

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- Reduced Genus : $\tilde{g} = \frac{1}{N}(g-1)$
- Theorem 1 : Let G be a non-abelian finite p-group of exponent p, where p is an odd prime. Let d be minimum number of generators of G and r_0 be its semi-rank. Then $\tilde{g} \geq 1$ is a reduced genus for G if and only if either :

  (1) $\tilde{g} = pg_0$ where $g_0 \geq d - r_0 - 1$ (Oesterlé)

  (2) $\tilde{g} + p - \frac{1}{2}(p-1)$ is expressible as $px + \frac{1}{2}(p-1)y$
  for $x, y \geq 0$, and such that $2x + y \geq d$.

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  for $x, y \geq 0$, and such that $2x + y \geq d$.
- Using these results it is possible to write down the explicit formulas for $\mu_0(G)$ and $\sigma_0(G)$

p-Groups of higher exponent

- Corollary to Thm 1 : The Genus Spectrum as described, is unique for p-groups of exponent p which have same order and which share the (graded) isomorphic first and second Pieces of the associated Lie Ring.

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- Theorem 2 : For p-groups of maximal class of order upto p^p there are precisely two Genus Spectrum in each order; one for exponent p and another for exponent p^2 (S.).

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- Theorem 2 : For p-groups of maximal class of order upto p^p there are precisely two Genus Spectrum in each order; one for exponent p and another for exponent p^2 (S.).
- We believe that the characerizing property of Genus Spectrum is connected to the isomorphism type of associated Graded Lie Ring.

Data Spectrum for finite p-groups

- Let G be a finite p-group of exponent p^e. We denote by $(h; x_1^{[p]}, \ldots, x_e^{[p^e]})$, a data with multiplicity, where entries are the multiplicities of the orders in the top.

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- Poset Structure on $\mathbb{D}(G)$:

$D \leq D'$ if :

(1) $h \leq h'$,

(2) $x_i \leq x_i'$ when $x_i \neq 0$ ($1 \leq i \leq e$).

(3) $x_i = 0, x_{i+1} \neq 0$ and $x_i' \neq 0$ ($1 \leq i \leq e-1$).

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  (3) $x_i = 0, x_{i+1} \neq 0$ and $x_i' \neq 0$ $(1 \leq i \leq e-1)$.
- If $D \in \mathbb{D}(G)$ then $\{D' : D \leq D'\} \subseteq \mathbb{D}(G)$.

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- Question : To find the minimal elements in $\mathbb{D}(G)$ where G is a p-group of maximal class. (Work in progress with Dietrich, Eick, Müller)
- We are still to find out which could be a better characterization for Genus Spectrum.