Resolutions for Bieberbach Groups using GAP and polymake

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What's it about?

Bieberbach Groups

Fundamental Domains

Resolutions

GAP, polymake and HAPcryst

Examples, Performance, Further Work

Bieberbach groups

Definition

Let $G \leq \mathrm{O}(n) \ltimes \mathbb{R}^n$ be discrete and cocompact group. Then G is called *crystallographic group*.
If G is torsion free, it is called *Bieberbach group*.

Facts

- (1^{st} Bieberbach) G contains a free abelian subgroup T of rank n and finite index (pure translations).
- G/T is called *point group* of G.
- $\mathbb{R}^n/G$ is a compact, flat Riemannian manifold.
- (3^{rd} Bieberbach) There is a 1–1 correspondence between Bieberbach groups and compact, flat Riemannian manifolds.

Idea

The structure of $\mathbb{R}^n/G$ is determined by the behaviour of G on a fundamental domain.
So find a fundamental domain and calculate a resolution from it.

Definition

Let G be a crystallographic group acting on $\mathbb{R}^n$. Any set $F \subseteq R^n$ which contains a system of G-orbit representatives R with $\overline{R} = F$ is called *fundamental domain* of G.

Fundamental Domains

Theorem (Dirichlet-Voronoi construction)

Let $x \neq y \in \mathbb{R}^n$. Set $\mathrm{H}(x,y) := \{\, a \in \mathbb{R}^n \mid \|x-a\| \leq \|y-a\| \}$. Let G be a crystallographic group and $x \in \mathbb{R}^n$ with $G_x = 1$. Then

$$\mathrm{D}(x, x^G) := \bigcap_{y \in x^G} \mathrm{H}(x,y)$$

is a fundamental domain of G.

The point group G/T is finite, so

- $D(x, x^G)$ is determined by only finitely many elements of x^G
- this fundamental domain is a polytope.

Cellular Resolution

Let $\mathfrak{P} = D(x, x^G) \subseteq \mathbb{R}^n$ be a fundamental domain of the Bieberbach group G. Let $\mathfrak{P}_i$ be the set of faces of dimension i of the tessellation of $\mathbb{R}^n$ by $\mathfrak{P}$. Using the natural boundary map and imposing some orientation on the faces, we get a chain complex

$$0 \to \mathfrak{P}_n \to \cdots \to \mathfrak{P}_1 \to \mathfrak{P}_0 \to 0$$

And as G is torsion free, we can identify faces with group elements and get a free $\mathbb{Z}G$ resolution of $\mathbb{Z}$:

$$0 \to (\mathbb{Z}G)^{k_n} \to \cdots \to (\mathbb{Z}G)^{k_1} \to (\mathbb{Z}G)^{k_0}$$

where k_i is the number of orbits of G on the i-dimensional faces (notice that $k_n = 1$).

Advantages

- Only finitely many terms of the resolution must be calculated
- Dimensions of modules tend to be smallish

Challanges

- Choice of starting point
- Works for Bieberbach groups only (faces are identified with group elements)
- Requires convex hull calculations

What is polymake?

1: Computational Geometry Software

- Free software written by Ewgenij Gawrilow and Michael Joswig (TU Berlin/TU Darmstadt).
- Does all sorts of computations with polytopes: convex hulls, combinatorial properties, visualization
- Has a simple command-line interface
- Supports programming via Perl scripting

2: A **GAP** Package

providing a simple interface to use polymake from within **GAP**.
Now **GAP** can calculate convex hulls!

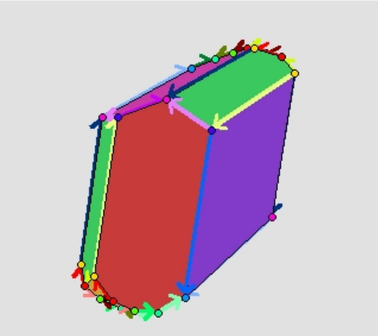
HAPcryst

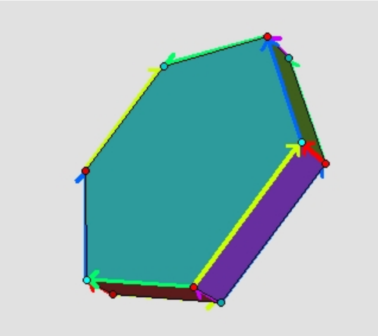
HAPcryst is an extension to Graham Ellis' HAP package. It does

- Orbit-Stabilizer like calculations for crystallographic groups
- Calculate fundamental domains of Bieberbach groups (using polymake)
- Calculate free resolutions of Bieberbach groups from fundamental domains
- Draw pictures (using JavaView)

Examples

`SpaceGroup(3,165)`, point group: C_6. Fundamental domains for starting points $(1/2, 0, 0)$ and $(0, 0, 0)$:





28 (12) vertices, 42 (18) edges and 16 (8) faces. Dimensions of modules in resolution: $7, 14, 8$ and $2, 5, 4$.

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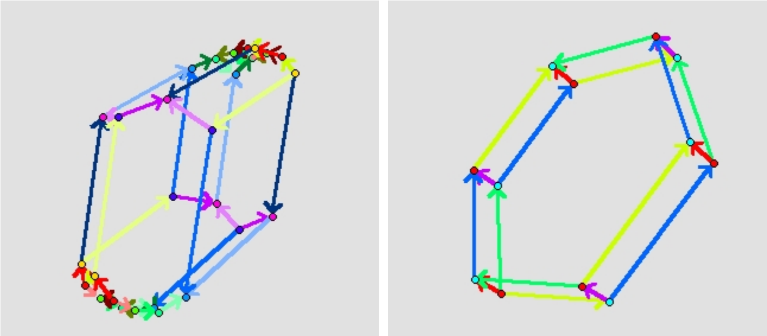

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Performance

Bieberbach group with point group $C_6 \times \mathrm{Alt}(4)$ acting on $\mathbb{R}^6$ (available from the CARAT website).
Calculate a free resolution with different starting points.

$(0, \ldots, 0)$ runtime: 42s (36s for **GAP**) dimensions: $20, 102, 194, 176, 79, 16, 1, 0, \ldots$

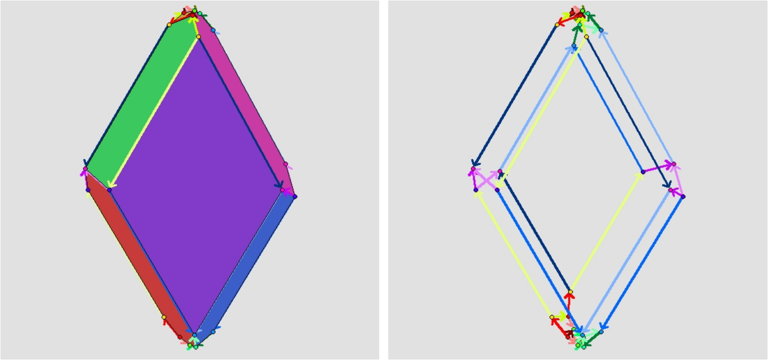
$(1/2, 1/3, 3/4, 1/5, 5/6, 1/7)$ runtime: 7h, dimensions: $873, 3259, 4574, 2963, 861, 87, 1, 0, \ldots$

The more general HAP function `ResolutionAlmostCrystalGroup` takes 23 hours to calculate 3 terms of a resolution with dimensions $1, 9, 39, 114$

Work to be done

- Calculate cohomology rings of Bieberbach groups.
- Calculate resolutions for non-Bieberbach groups
- Produce a nice result to prove usefulness.

A fundamental domain



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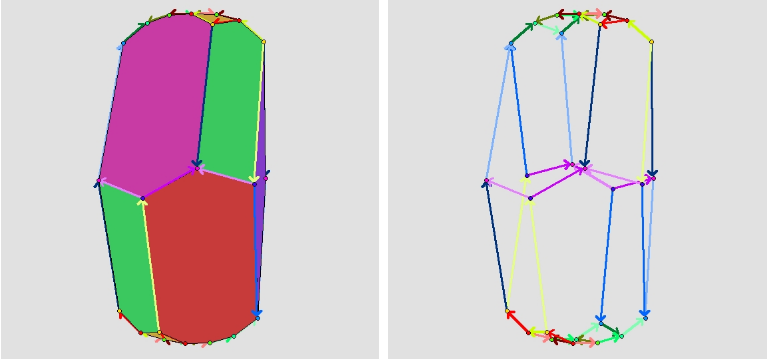

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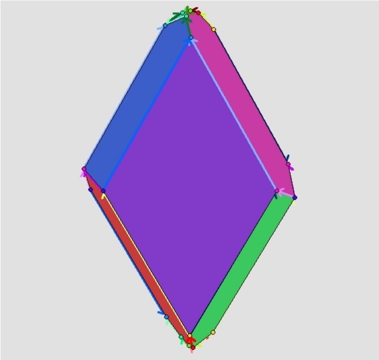

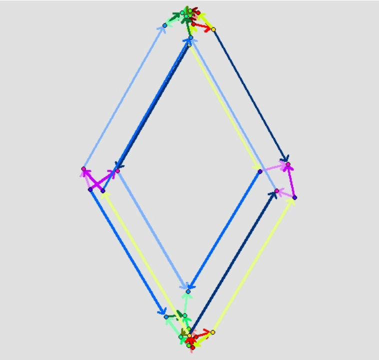

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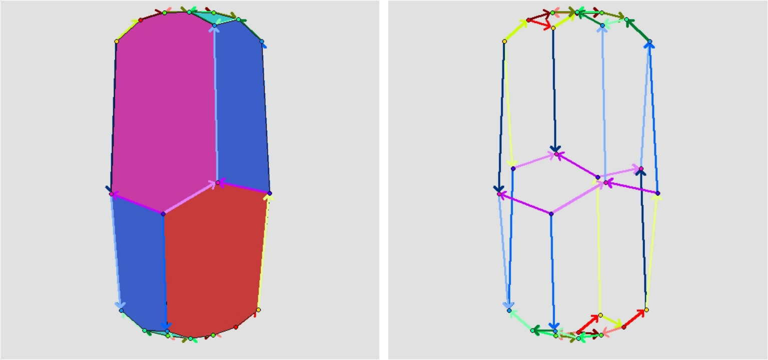

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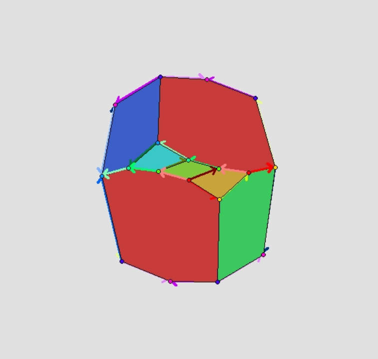 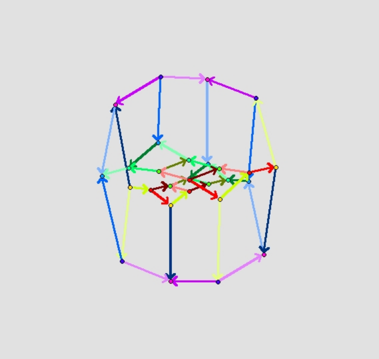

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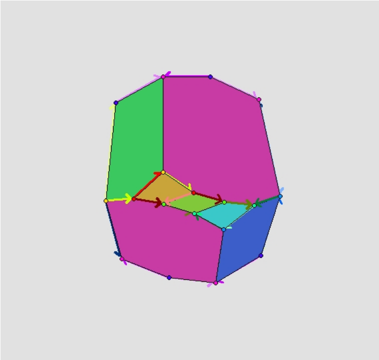
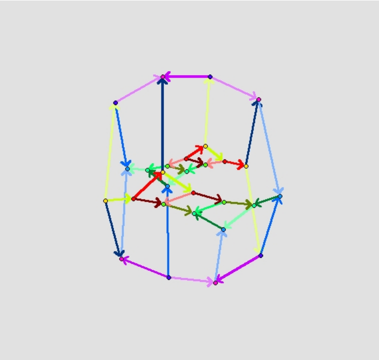
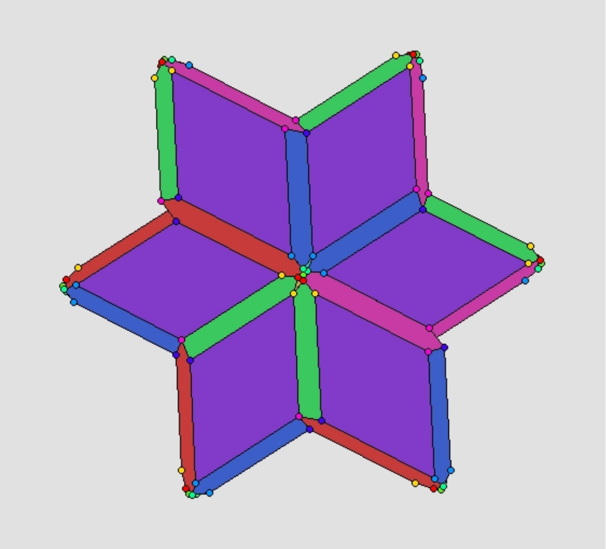




And the tessellation



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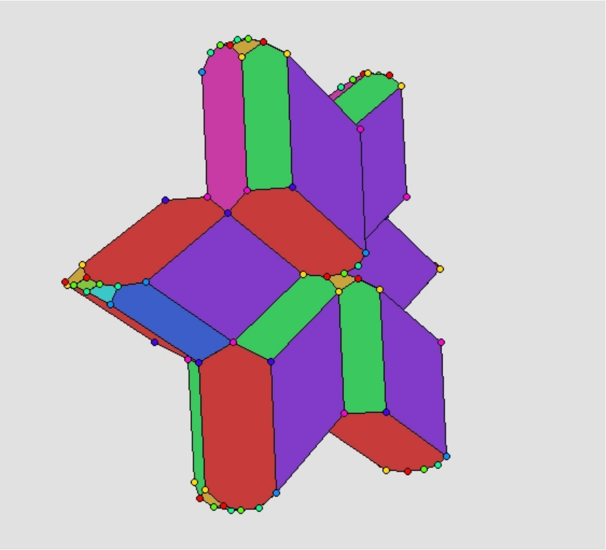

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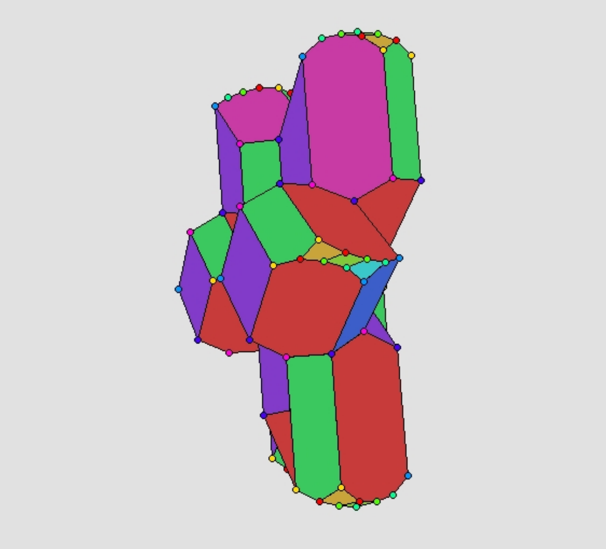

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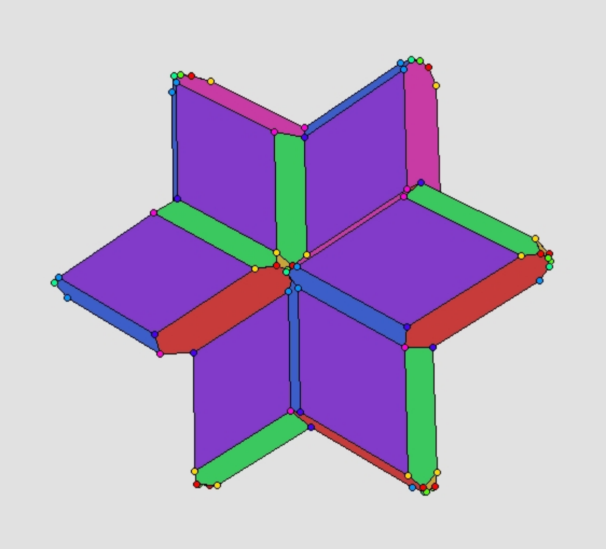

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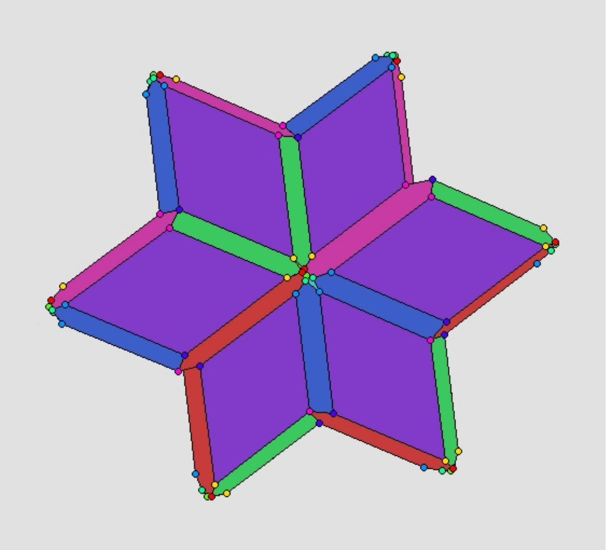

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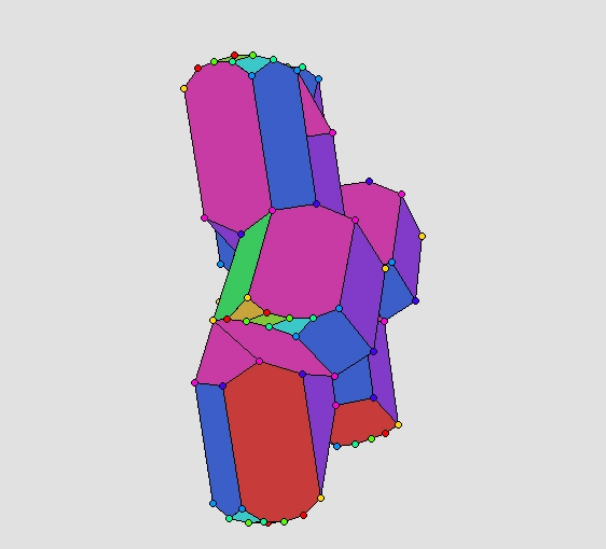

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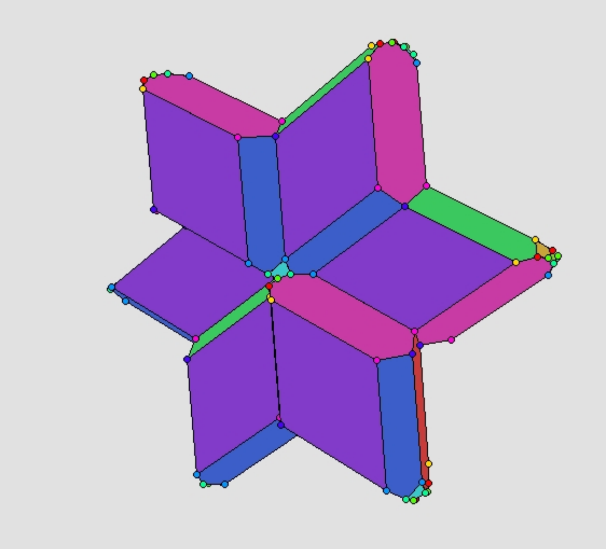

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