# Enumerating big orbits and an application: $B$ acting on the cosets of $Fi_{23}$

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[J.M.-Neunhöffer-Wilson]

J. Algebra 314, 2007, 75–96

http://www.math.rwth-aachen.de/∼Juergen.Mueller

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[J.M.-Neunhöffer-Noeske]

GAP-4 package ORB
(Version 0.999. . .)

http://www.math.rwth-aachen.de/
∼Max.Neunhoeffer/Computer/Software/Gap/orb.html

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# Multiplicity-free actions

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[Breuer-J.M., $\leq$ 2006]

Character tables of endomorphism rings of multiplicity-free permutation modules of the sporadic simple groups and their cyclic and bicyclic extensions

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• Contain information on the associated orbital graphs: distance-transitivity and -regularity, Ramanujan property.

• Classification of multiplicity-free actions: [Breuer-Lux, 1996; Linton-Mpono, $\sim$ 2001; Breuer, 2005].

• Building on earlier work (amongst others): [Ivanov-Linton-Lux-Saxl-Soicher, 1995; Praeger-Soicher, 1997; Höhler, 2001; ...].

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## BIG orbits — multiplicity-free actions of $B$:

• on the 13 571 955 000 cosets of $2.^2E_6(2).2$ [Higman, 1976],

• on the 27 143 910 000 cosets of $2.^2E_6(2)$ [Higman, 1976],

• on the 11 707 448 673 375 cosets of $2^{1+22}.Co_2$ [J.M., 2003; Rowley-Walker, 2004],

• on the 1 015 970 529 280 000 cosets of $Fi_{23}$ [J.M.-Neunhöffer-Wilson, 2007].

# $B$ acting on the set $X$ of cosets of $H := Fi_{23}$

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- 23 orbits $X_i = x_i H \subseteq X$,

neither $k_i := |X_i|$ nor $H_i := \mathrm{Stab}_H(x_i)$ were known before.

- $H$-orbit representatives $x_i$ were found by random search,

and using the action of the Monster $\mathbb{M}$ on 6-transpositions.

| $i$ | $k_i$ | $\|H_i\|$ | $H_i$ |
|---:|---:|---:|---:|
| 1 | 1 | $\sim 4.1 \cdot 10^{18}$ | $Fi_{23}$ |
| 2 | 412 896 | 9 904 359 628 800 | $O_8^+(3){:}\,2_2$ |
| 3 | 86 316 516 | 47 377 612 800 | $S_8(2)$ |
| 4 | 195 747 435 | 20 891 566 080 | $2^{11}.M_{23}$ |
| 5 | 8 537 488 128 | 479 001 600 | $S_{12}$ |
| 6 | 23 478 092 352 | 174 182 400 | $O_8^+(2)$ |
| 7 | 33 816 182 400 | 120 932 352 | $[3^9].[2^{10}].S_3$ |
| 8 | 113 778 447 552 | 35 942 400 | $2 \times {}^2F_4(2)'$ |
| 9 | 160 533 964 800 | 25 474 176 | $S_3 \times G_2(3)$ |
| 10 | 504 245 392 560 | 8 110 080 | $2^{10}.M_{11}$ |
| 11 | 1 044 084 577 536 | 3 916 800 | $S_4(4){:}\,4$ |
| 12 | 1 152 560 897 280 | 3 548 160 | $(2 \times 2.M_{22}).2$ |
| 13 | 1 584 771 233 760 | 2 580 480 | $2^7.A_8$ |
| 14 | 5 282 570 779 200 | 774 144 | $2^7.U_3(3)$ |
| 15 | 7 888 639 030 272 | 518 400 | $(A_6 \times A_6){:}\,2^2$ |
| 16 | 12 678 169 870 080 | 322 560 | $2^2.L_3(4).2^2$ |
| 17 | 21 514 470 082 560 | 190 080 | $2 \times M_{12}$ |
| 18 | 43 028 940 165 120 | 95 040 | $M_{12}$ |
| 19 | 50 712 679 480 320 | 80 640 | $2.L_3(4).2_2$ |
| 20 | 133 120 783 635 840 | 30 720 | $2^4.2^4.A_5.2$ |
| 21 | 190 172 548 051 200 | 21 504 | $2^6{:}\,L_3(2){:}\,2$ |
| 22 | 262 954 634 342 400 | 15 552 | $3^4.2^{1+4}.S_3$ |
| 23 | 283 991 005 089 792 | 14 400 | $(A_5 \times A_5){:}\,2^2$ |

# Orbit enumeration by suborbits

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**Basic idea:**
[Parker-Wilson, ∼1995; Lübeck-Neunhöffer, 2001; O'Brien; Kerber-Kohnert-Laue]

• Let $X$ be a $G$-set, let $U < G$ be a **helper subgroup**, let $Y$ be a $U$-set and let $\bar{\ }\colon X \to Y$ be a $U$-homomorphism.

**Common case:** $X \subseteq M$, where $M$ is an $\mathbb{F}G$-module, and $\bar{\ }$ is induced by an $\mathbb{F}U$-module homomorphism from $M|_U$.

• For any $U$-orbit in $Y$ designate a $U$**-minimal** point in it, and $x \in X$ is called $U$**-minimal** if $\overline{x} \in Y$ is $U$-minimal.

• Enumerate $X$ by $U$-orbits, store the $U$-minimal points in $X$.

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**Orbit-stabiliser by $U$-orbits on $X = x_1 G$:**

**a)** Procedure $\mathsf{Minimaliser}_U(x)$:
For $x \in X$ compute $u \in U$ such that $xu$ is $U$-minimal.

**b)** Procedure $\mathsf{BarStabiliser}_U(x)$:
For $U$-minimal $x \in X$ compute $\overline{S} := \mathrm{Stab}_U(\overline{x})$ and $|\overline{S}|$.

• Applying $\mathsf{Minimaliser}_U(x)$ allows to check whether a $U$-orbit has been encountered earlier.

∘ If not, the $U$-minimal points in $xU$ are computed by an orbit-stabiliser algorithm using $\overline{S} = \mathsf{BarStabiliser}_U(x)$.
∘ Otherwise, collect elements of $\mathrm{Stab}_G(x_1)$.

• Assume that **orders of subgroups** generated by subsets of $G$ can be efficiently determined.

# Iterating orbit enumeration by suborbits

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• Let $V < U < G$ be helper subgroups, and compute a **left transversal $\mathcal{L}$ of $V$ in $U$**.

• Let $Z$ be a $V$-set, let $\widetilde{\ }: Y \to Z$ be a $V$-homomorphism, and assume $\mathsf{Minimaliser}_V(y)$ and $\mathsf{BarStabiliser}_V(y)$ to be given, hence the $U$-orbits in $Y$ can be enumerated by $V$-orbits.

• For any $U$-orbit in $Y$ designate a **$U$-minimal** point $y$ **amongst the $V$-minimal points** in it. **Moreover:**

∘ For any $V$-minimal $y' \in yU \setminus yV$ find $u \in \mathcal{L}$ such that $y'u \in yV$.

∘ For any $V$-minimal $y' \in yV$ find $v \in \mathrm{Stab}_V(\widetilde{y}) = \mathsf{BarStabiliser}_V(y)$ such that $y'v$ is $U$-minimal.

**a)** Procedure $\mathsf{Minimaliser}_U(x)$:
Let $v' := \mathsf{Minimaliser}_V(\overline{x})$, hence $\overline{x}v' \in yU$ is $V$-minimal.
Hence $\overline{x}v'u \in yV$, where $y$ is $U$-minimal.
Let $v'' := \mathsf{Minimaliser}_V(\overline{x}v'u)$, hence $\overline{x}v'uv'' \in yV$ is $V$-minimal.
Thus $\overline{x}v'uv''v = y$, and $\mathsf{Minimaliser}_U(x) := v'uv''v$.

**b)** Procedure $\mathsf{BarStabiliser}_U(x)$:
$\mathrm{Stab}_U(\overline{x})$ is found by enumerating the $U$-orbit $\overline{x}U$ by $V$-orbits.

• **Iteration** for chains $\{1\} = U_0 < U_1 < U_2 < \cdots < U_k < G$.

• For $U_1$ every point is $U_0$-minimal, and $\mathsf{Minimaliser}_{U_0}$ and $\mathsf{BarStabiliser}_{U_0}$ are trivial.

# In practice

- Index $|X| = [B\colon Fi_{23}] = 1\,015\,970\,529\,280\,000 \sim 1.0 \cdot 10^{15}$, realizing $X \subseteq \mathbb{F}_2^{4371}$ needs $\lceil 4371/8 \rceil = 547$ Bytes per vector, needs $555\,735\,879\,516\,160\,000 \sim 5.6 \cdot 10^{17}$ Bytes for all of $X$.

- The subgroup chain:

| $j$ | $U_j$ | $\lvert U_j \rvert$ | $[U_j\colon U_{j-1}]$ | $\dim_{\mathbb{F}_2}(M_j)$ |
|---|---|---|---|---|
| 5 | $B$ | $\sim 4.2 \cdot 10^{33}$ | $\sim 1.0 \cdot 10^{15}$ | 4371 |
| 4 | $Fi_{23}$ | $\sim 4.1 \cdot 10^{18}$ | 86 316 516 | 782 |
| 3 | $S_8(2)$ | 47 377 612 800 | 2 295 | 42 |
| 2 | $2^{10}\colon A_8$ | 20 643 840 | 8 192 | 31 |
| 1 | $A_7$ | 2 520 | 2 520 | 18 |

- Enumeration of the $H$-orbits in $X_i^\pi \subseteq M_4$ by $U_3$-orbits.

- Apply group generators to $U_3$-orbit representatives only.

- Ignore points $x$ such that $|\mathrm{Stab}_{U_3}(\overline{x})| > 10^5$.

- Needs $\sim 4\,800$ s $\sim 80$ min of CPU time on a 3.2 GHz Pentium IV, and $\sim 1.1 \cdot 10^9$ Bytes.

| $i$ | $k_i^\pi = \lvert X_i^\pi \rvert$ | $\lvert \mathcal{X}_i \rvert$ | $k_i^\pi / \lvert \mathcal{X}_i \rvert$ | $U_3$-orbits | $N_i$ | $\lvert \mathcal{X}_i \rvert / N_i$ |
|---:|---:|---:|---:|---:|---:|---:|
| 23 | 283 991 005 089 792 | 281 092 626 984 960 | 0.99 | 8 109 | 1 430 821 | 196 455 480 |
| 22 | 262 954 634 342 400 | 259 808 546 995 200 | 0.99 | 6 968 | 1 212 147 | 214 337 491 |
| 21 | 190 172 548 051 200 | 187 996 976 179 200 | 0.99 | 5 256 | 1 227 646 | 153 136 145 |
| 20 | 133 120 783 635 840 | 131 937 126 773 760 | 0.99 | 3 936 | 601 292 | 219 422 721 |
| 19 | 50 712 679 480 320 | 49 876 298 933 760 | 0.98 | 1 930 | 254 398 | 196 056 175 |
| 18 | 43 028 940 165 120 | 42 270 766 755 840 | 0.98 | 1 476 | 447 462 | 94 467 836 |
| 17 | 21 514 470 082 560 | 21 046 385 848 320 | 0.98 | 769 | 211 988 | 99 281 025 |
| 16 | 12 678 169 870 080 | 12 380 329 543 680 | 0.98 | 535 | 149 079 | 83 045 429 |
| 15 | 7 888 639 030 272 | 7 687 679 811 840 | 0.97 | 504 | 78 843 | 97 506 180 |
| 14 | 2 641 285 389 600 | 2 566 592 870 400 | 0.97 | 155 | 69 000 | 37 196 998 |
| 13 | 1 584 771 233 760 | 1 495 816 519 800 | 0.94 | 136 | 99 628 | 15 014 017 |
| 12 | 1 152 560 897 280 | 1 087 842 631 680 | 0.94 | 101 | 25 699 | 42 330 154 |
| 11 | 1 044 084 577 536 | 1 010 524 999 680 | 0.97 | 97 | 15 298 | 66 056 020 |
| 10 | 252 122 696 280 | 222 345 768 960 | 0.88 | 33 | 9 029 | 24 625 736 |
| 6 | 23 478 092 352 | 19 780 035 840 | 0.84 | 20 | 3 808 | 5 194 337 |
| 7 | 11 272 060 800 | 9 952 588 800 | 0.88 | 7 | 1 556 | 6 396 265 |
| 5 | 8 537 488 128 | 7 229 107 200 | 0.85 | 10 | 965 | 7 491 303 |
| 9 | 148 642 560 | 135 080 640 | 0.91 | 5 | 17 794 | 7 591 |
| 2 | 412 896 | 366 792 | 0.89 | 2 | 122 | 3 006 |
| 8 | 31 671 | 18 360 | 0.58 | 1 | 8 | 2 295 |

# The character table of $B$ on $Fi_{23}$

| $\varphi$ | $\chi_\varphi$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 412896 | 86316516 | 195747435 | 8537488128 | 23478092352 | 33816182400 | 113778447552 |
| 2 | 4371 | 1 | −137632 | 18115812 | −10472085 | −1159411968 | 1449264960 | 3757353600 | 1404672192 |
| 3 | 96255 | 1 | 82016 | 8890596 | 5701995 | 457037568 | 327742272 | 1297296000 | −1788671808 |
| 4 | 9458750 | 1 | 41888 | 3232548 | −43605 | 123026688 | 57841344 | 314160000 | 183218112 |
| 5 | 63532485 | 1 | −32032 | 2275812 | 414315 | −77223168 | −2312640 | 179625600 | −32332608 |
| 6 | 347643114 | 1 | 10208 | 704484 | 1589355 | 10679040 | 46398528 | −9609600 | 57081024 |
| 7 | 356054375 | 1 | −17248 | 900900 | −1508949 | −20097792 | 43902144 | 32672640 | −21155904 |
| 8 | 4221380670 | 1 | −3232 | 324324 | 103275 | −2453760 | 15121728 | −12297600 | −15494976 |
| 9 | 4275362520 | 1 | 14816 | 725796 | −43605 | 16743168 | −7316928 | 31920000 | 14841792 |
| 10 | 9287037474 | 1 | 6896 | 132516 | 699435 | 736128 | 11096352 | 4502400 | −38864448 |
| 11 | 13508418144 | 1 | −11632 | 475812 | 111915 | −9283968 | −491040 | 17673600 | 7584192 |
| 12 | 108348770530 | 1 | 7328 | 246564 | −43605 | 3421440 | 1729728 | 4502400 | −11866176 |
| 13 | 309720864375 | 1 | −1120 | 89892 | −181845 | −172800 | 3172032 | −3638400 | 6934464 |
| 14 | 635966233056 | 1 | 3408 | 69284 | 147755 | 295040 | 2450528 | −169600 | 6681024 |
| 15 | 1095935366250 | 1 | −4576 | 126756 | 2475 | −1324800 | −949824 | 1061760 | −254016 |
| 16 | 6145833622500 | 1 | 2864 | 51876 | −26325 | 316800 | −507744 | 309120 | 1197504 |
| 17 | 6619124890560 | 1 | 1088 | 39204 | 25515 | 138240 | −300672 | −1065600 | −1498176 |
| 18 | 12927978301875 | 1 | −2128 | 19620 | −40149 | 67968 | 706464 | 186240 | −627264 |
| 19 | 38348970335820 | 1 | −1232 | 15524 | 37675 | 19840 | −69472 | −233600 | −576 |
| 20 | 89626740328125 | 1 | 944 | 1188 | 15147 | −79488 | 61344 | 63360 | 36288 |
| 21 | 211069033500000 | 1 | 560 | 1188 | −12501 | −51840 | 12960 | −68736 | −129600 |
| 22 | 284415522641250 | 1 | −16 | −5724 | 8235 | 17280 | 50976 | 78720 | −46656 |
| 23 | 364635285437500 | 1 | −400 | −1116 | −5589 | 26496 | −71136 | −7296 | 119232 |

| 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|
| 160533964800 | 504245392560 | 1044084577536 | 1152560897280 | 1584771233760 | 5282570779200 | 7888639030272 | 12678169870080 |
| −5945702400 | 39426594480 | −21483221760 | −4743048960 | −110868769440 | 65216923200 | −292171815936 | 573908924160 |
| −511948800 | 12027702960 | −9527341824 | 6966984960 | 30484602720 | 28447848000 | 58091185152 | 118446831360 |
| 258508800 | 1991288880 | 1252323072 | −1021697280 | 4906012320 | −3514104000 | 3727696896 | 12802648320 |
| 35481600 | 1084693680 | 550851840 | −432034560 | −2400567840 | 1235995200 | −300174336 | 4718165760 |
| −167270400 | 224426160 | 533820672 | 271607040 | −9741600 | 916660800 | 2067158016 | −1656357120 |
| 63866880 | 185985072 | −186810624 | 778242816 | −259829856 | −2109032640 | −1909619712 | −643458816 |
| 74188800 | 87499440 | −219034368 | −142145280 | 29121120 | 499867200 | −274627584 | −544631040 |
| 4147200 | 110118960 | −61012224 | 62588160 | 198033120 | 197640000 | −366363648 | 5218560 |
| 20044800 | −21727440 | 115105536 | 171953280 | 32315760 | 217339200 | −118153728 | 122446080 |
| −18662400 | 32946480 | −61205760 | −22584960 | −74323440 | −10756800 | 200600064 | −34179840 |
| −6912000 | 5609520 | −1790208 | −28857600 | −1265760 | −80222400 | 35030016 | −96802560 |
| −6912000 | 12798000 | 19554048 | −7568640 | 3745440 | −43200 | −48356352 | −17729280 |
| 5913600 | −1900240 | −8656128 | 8992640 | −2385200 | −15211200 | 36246016 | 7220480 |
| 1935360 | −841680 | 6983424 | 3168000 | 10755360 | 2721600 | 1741824 | −31921920 |
| 691200 | −2857680 | 2467584 | −777600 | −4879440 | 5417280 | −5515776 | 518400 |
| −460800 | 2430000 | −1928448 | 3732480 | −3810240 | 648000 | 5308416 | 933120 |
| −414720 | −2332368 | −1292544 | −307584 | −943056 | 2928960 | 787968 | 6269184 |
| 76800 | −292560 | 472832 | −1668480 | 588720 | −1924800 | −2025984 | 4348160 |
| −709632 | −452304 | −850176 | 134784 | 854064 | 938304 | −1866240 | 518400 |
| 248832 | 73008 | 200448 | −335232 | 518832 | −720576 | 898560 | 1237248 |
| 138240 | 114480 | 532224 | −293760 | −481680 | 25920 | −262656 | −1416960 |
| −82944 | 86832 | −352512 | 508032 | −42768 | 191808 | 290304 | −311040 |

| 17 | 18 | 19 | 20 | 21 | 22 | 23 |
|---|---|---|---|---|---|---|
| 21514470082560 | 43028940165120 | 50712679480320 | 133120783635840 | 190172548051200 | 262954634342400 | 283991005089792 |
| −796832225280 | 531221483520 | 1460859079680 | −2739110774400 | −782603078400 | 3246353510400 | −1168687263744 |
| 158430504960 | −222361251840 | 239651343360 | 190079809920 | −857327328000 | 28598169600 | 218194808832 |
| 10166446080 | 20332892160 | 7936220160 | 8210885760 | 47791814400 | −25333862400 | −90188550144 |
| −4534548480 | −8511713280 | −1053803520 | 12753417600 | 10828857600 | −17953689600 | 3908653056 |
| −679311360 | 1892782080 | 3994721280 | −5895711360 | 1568160000 | −10005811200 | 6838013952 |
| 1675634688 | 1177473024 | 3238050816 | −155675520 | −44478720 | −6826659840 | 4981616640 |
| −5806080 | 592220160 | 722856960 | 813214080 | −13996800 | −1025740800 | −578285568 |
| −75479040 | −452874240 | −1233239040 | −1778474880 | 666144000 | 148377600 | 2518290432 |
| −322237440 | 661893120 | −489991680 | 959091840 | −1020988800 | 174182400 | −479582208 |
| 269982720 | 836075520 | −664312320 | −183254400 | −1004918400 | 593510400 | 125024256 |
| −145152000 | −11612160 | 83082240 | 268168320 | −170553600 | 212889600 | −59609088 |
| 18524160 | −16035840 | −61793280 | 98133120 | −116640000 | 190771200 | −74649600 |
| −39797760 | −41656320 | −22725120 | 16717440 | 9264000 | 80076800 | −41576448 |
| −5806080 | −58060800 | 36449280 | −18264960 | 41644800 | 94187520 | −83349504 |
| 14515200 | 11612160 | 15137280 | 9797760 | −15085440 | −21934080 | −10450944 |
| 14100480 | −9953280 | −9953280 | −18195840 | 27993600 | 27648000 | −35831808 |
| 7216128 | −6967296 | −2225664 | −16744320 | 22654080 | −8663040 | −276480 |
| −1582080 | 5468160 | −919040 | −1537920 | −7036800 | −17100800 | 23365632 |
| −746496 | −1658880 | 2198016 | 3825792 | 6065280 | −4534272 | −3815424 |
| −1410048 | 995328 | −1893888 | −4053888 | −1316736 | −2764800 | 8570880 |
| 2903040 | −1658880 | −69120 | −3058560 | 51840 | 6082560 | −2709504 |
| −1741824 | 995328 | 705024 | 4572288 | −1026432 | −700416 | −3151872 |