A Lemma of C.T.C. Wall

Braunschweig, September 2007

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Contents

1. DEFINITIONS

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- *(Freeness)* R_n is a free A-module for all $n \geq 0$,
- *(Augmentation)* the cokernel of d_1 is isomorphic to the module M.

For A-modules M, N define

$$Ext_A^n(M,N) = \frac{\ker(Hom_A(R_n, N) \to Hom_A(R_{n+1}, N))}{\mathrm{image}(Hom_A(R_{n-1}, N) \to Hom_A(R_n, N))}$$

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and

$$Tor_n^A(M,N) = \frac{\ker(R_n \otimes_A N \to R_{n-1} \otimes_A N)}{\mathrm{image}(R_{n+1} \otimes_A N \to R_n \otimes_A N)}.$$

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My motivation is not:

A system is controllable if one can move from one system trajectory x_0 to another trajectory x_1 without violating the system law. Some systems are more controllable than others.

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For an A-module M (arising from a system) set $N = Hom_A(M, A)$.

Definition

The controllability degree of M is the first natural number $n > 0$ such that $Ext^n_A(N, A) \neq 0$ and $Ext^i_A(N, A) = 0$ for $0 < i < n$.

The are a number of packages for computing these functors.

CoCoA, Macaulay, and Singular contain a range of Gröbner basis methods for computing the functors $Tor_n^A(M,N)$ and $Ext_A^n(M,N)$ in the case where k is a field and the ring A is commutative.

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The Plural extension to Singular handles certain non-commutative rings A.

For the cohomology of a group G one takes the ring of integers $k = \mathbb{Z}$, the module $M = \mathbb{Z}$ with trivial G-action, the group ring $A = \mathbb{Z}G$, and sets

$$H^n(G, N) = Ext^n_{\mathbb{Z}G}(\mathbb{Z}, N), \quad H_n(G, N) = Tor_n^{\mathbb{Z}G}(\mathbb{Z}, N).$$

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MAGMA handles $n > 2$ for G a small p-group (where it suffices to set $k = GF(p)$).

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The computation of cohomology involves two expensive but independent tasks:

1. the computation of a free resolution;
2. the computation of the homology of a chain complex.

This talk focuses on a method for task 1.

2. EXAMPLE COMPUTATIONS

Theorem

(i) The group $K_3 = \ker(\mathrm{SL}_2(\mathbb{Z}_{3^3}) \to \mathrm{SL}_2(\mathbb{Z}_3))$ *has third integral homology group of exponent 27.*

(ii) In dimensions $n \neq 3$, $1 \leq n$ *it has integral homology* $H_n(K_3, \mathbb{Z})$ *of exponent at most 9.*

Proof.

(i) W. Browder and J. Pakianathan, "Cohomology of uniformly powerful p-groups", *Trans. Amer. Math. Soc.* 352 (2000), no. 6, 2659–2688.

(ii) J. Pakianathan, "Exponents and the cohomology of finite groups", *Proc. Amer. Math. Soc.* 128 (2000), no. 7, 1893–1897.

Theorem

(i) The group $K_3 = \ker(\mathrm{SL}_2(\mathbb{Z}_{3^3}) \to \mathrm{SL}_2(\mathbb{Z}_3))$ *has third integral homology group of exponent 27.*

(ii) In dimensions $n \neq 3$, $1 \leq n \leq 6$ *it has integral homology* $H_n(K_3, \mathbb{Z})$ *of exponent at most 9.*

Automated Proof.

```
gap> K3:=MaximalSubgroups(SylowSubgroup(
                            SL(2,Integers mod 3^3),3))[2];;
gap> K3:=Image(IsomorphismPcGroup(K3));;
gap> Display(List([1..4],n->GroupHomology(K3,n)));
[ [   3,   3,   3 ],
  [   3,   3,   3 ],
  [   3,   3,   3,   3,   3,   3,  27 ],
  [   3,   3,   3,   3,   3,   3,   3,   3 ],
  [   3,   3,   3,   3,   3,   3,   3,   3,   3,
      3,   9,   9,   9 ],
  [   3,   3,   3,   3,   3,   3,   3,   3,   3,
      3,   9,   9,   9,   9,   9 ] ]
```

Theorem

The Mathieu group M_{23} has trivial integral homology $H_n(M_{23}, \mathbb{Z}) = 0$ in dimensions $n = 1, 2, 3$.

Proof.

R.J. Milgram, "The cohomology of the Mathieu group M_{23}", *J. Group Theory* 3 (2000), no. 1, 7–26.

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Automated Proof.

```
gap> GroupHomology(MathieuGroup(23),1);
[  ]
gap> GroupHomology(MathieuGroup(23),2);
[  ]
gap> GroupHomology(MathieuGroup(23),3);
[  ]
```

Theorem

The mod 2 cohomology $H^n(M_{11}, \mathbb{Z}_2)$ of the Mathieu group M_{11} is a vector space of dimension equal to the coefficients of x^n in the Poincaré series

$$(x^4 - x^3 + x^2 - x + 1)/(x^6 - x^5 + x^4 - 2x^3 + x^2 - x + 1)$$

for all n.

Proof.

P.J. Webb, "A local method in group cohomology" *Comment. Math. Helv.* 62 (1987), no. 1, 135–167.

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$$(x^4 - x^3 + x^2 - x + 1)/(x^6 - x^5 + x^4 - 2x^3 + x^2 - x + 1)$$

for all $n \leq 20$.

Automated Proof.

```
gap> PoincareSeriesPrimePart(MathieuGroup(11),2,20);
(x^4-x^3+x^2-x+1)/(x^6-x^5+x^4-2*x^3+x^2-x+1)
```

e t c .

3. CONTRACTING HOMOTOPIES

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$$\longrightarrow R_n \xrightarrow{d_n} R_{n-1} \longrightarrow \cdots \longrightarrow R_0$$

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- **R!.dimension(n)** $= \mathrm{Rank}_{\mathbb{Z}G} R_n$
- **R!.boundary(n,k)** $=$ a list of integer pairs representing the boundary of the kth free generator in degree n
- **And more . . .**

The following element of choice occurs frequently in homological algebra:

For each $x \in \ker(d_n \colon R_n \to R_{n-1})$ *choose an element* $\tilde{x} \in R_{n+1}$ *such that* $d_{n+1}(\tilde{x}) = x$*.*

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R!.homotopy(n,[i,g]) = contracting homotopy

If G is large or infinite one can't solve $d_{n+1}(\tilde{x}) = x$ using basic linear algebra over $\mathbb{Z}$.

A contracting homotopy on R_* is a family of abelian group homomorphisms $h_n \colon R_n \to R_{n+1}$ ($n \geq 0$) satisfying

$$d_{n+1} h_n(x) + h_{n-1} d_n(x) = x$$

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Lemma

Setting $\tilde{x} = h_n(x)$ ensures $d_{n+1}(\tilde{x}) = x$.

So we need a range of methods for providing contracting homotopies.

Geometry can provide resolutions with contracting homotopy.

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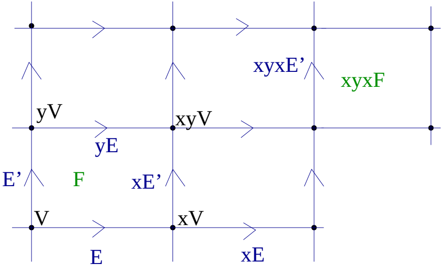
There is a G-**equivariant** cellular decomposition of $X = \mathbb{R}^2$.



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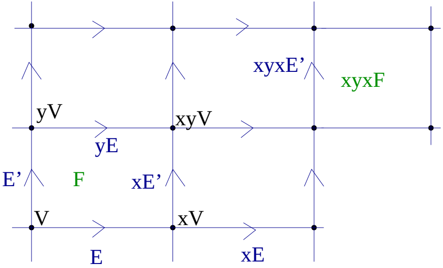

$C_n(X)$ = free abelian group on n-cells.
X contractible $\Rightarrow$ cellular chain complex $C_*(X)$ is exact.

$$C_*(X): \quad \cdots \to C_3(X) \xrightarrow{d_3} C_2(X) \xrightarrow{d_2} C_1(X) \xrightarrow{d_1} C_0(X)$$

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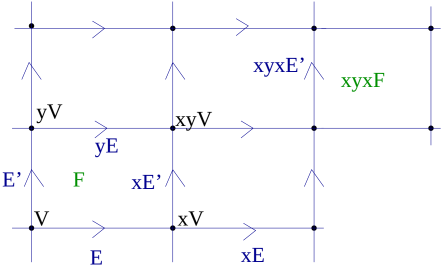

We view

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as a chain complex of $\mathbb{Z}G$-modules:

$$\cdots \to 0 \xrightarrow{d_3} \mathbb{Z}G \xrightarrow{d_2} \mathbb{Z}G \oplus \mathbb{Z}G \xrightarrow{d_1} \mathbb{Z}G$$

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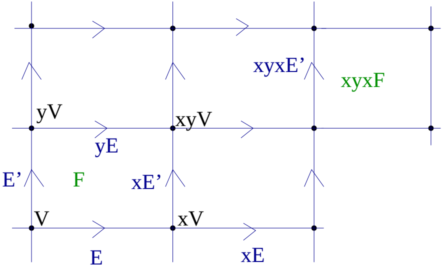

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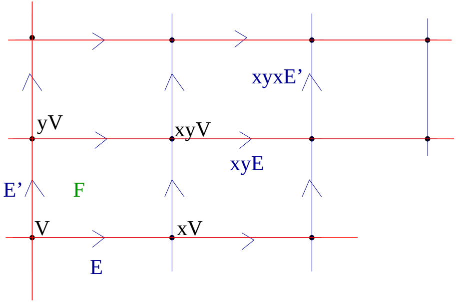
$$d_2(F) = (1-x)E' + (y-1)E$$

A homotopy homomorphism $h_0\colon C_0(X) \to C_1(X)$ can be specified by setting $Y^0 = \{V\}$ and choosing a maximal contractible cellular subspace Y^1 in the 1-skeleton.



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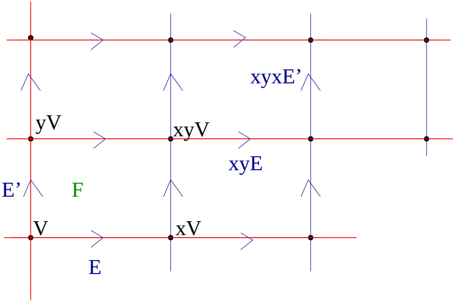

Now, for example

$$h_0(xyV) = E' + yE$$

since this corresponds to the path from V to xyV in Y^1.

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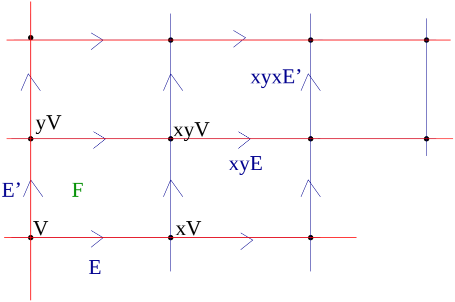

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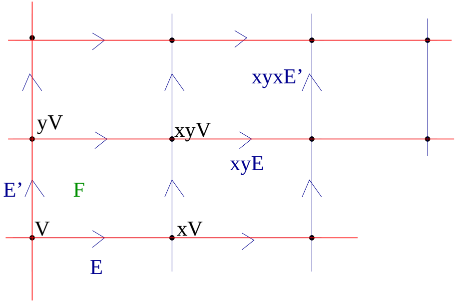
$$h_0(V) = 0$$

since $V \in Y^0$.

A homotopy homomorphism $h_1\colon C_1(X) \to C_2(X)$ can be specified by choosing a maximal contractible cellular subspace Y^1 in the 1-skeleton and a maximal contractible cellular subspace Y^2 of the 2-skeleton.



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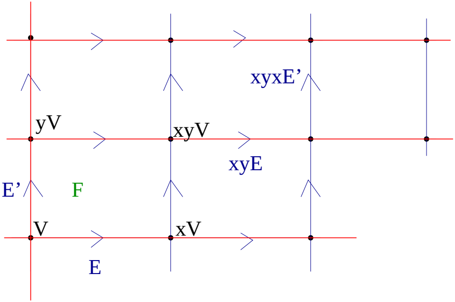

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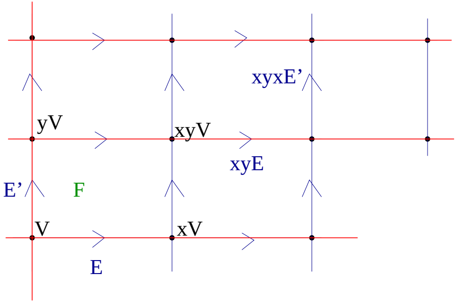

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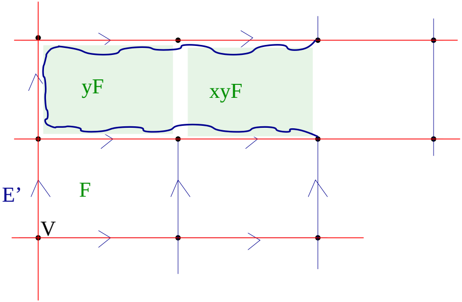

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4. THE LEMMA

Let A be a ring. Let

$$C_*\colon \to C_n \to C_{n-1} \to \cdots \to C_0$$

be an A-resolution of some A-module M, where the A-modules C_n are **NOT assumed to be free.**

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Suppose that, for each m, we have a free A-resolution of C_m

$$D_{m*}\colon \to D_{m,n} \to D_{m,n-1} \to \cdots \to D_{m,0} \to C_m$$

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Lemma (C.T.C. Wall)

There exists a free A-resolution $R_* \to M$ with

$$R_n = \bigoplus_{p+q=n} D_{p,q}.$$

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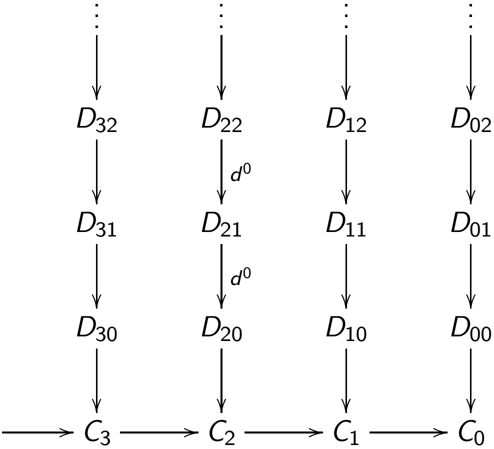
Lemma (C.T.C. Wall)

There exists a free A-resolution $R_* \to M$ with

$$R_n = \bigoplus_{p+q=n} D_{p,q}.$$

The proof can be made constructive using the notion of contracting homotopy. Furthermore, one can derive an explicit formula for a contracting homotopy on R_* in terms of contracting homotopies on the D_{m*} and C_*.



$$d^0 d^0 = 0$$

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & D_{32} & \longrightarrow & D_{22} & \longrightarrow & D_{12} & \longrightarrow & D_{02} \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & D_{31} & \longrightarrow & D_{21} & \longrightarrow & D_{11} & \longrightarrow & D_{01} \\
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$$

$$\partial = d^0 + d^1$$

$$
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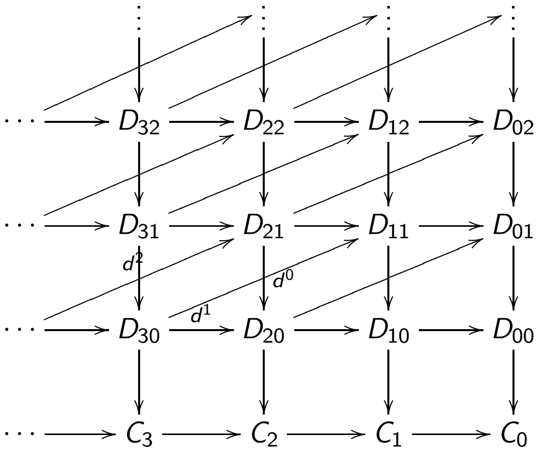
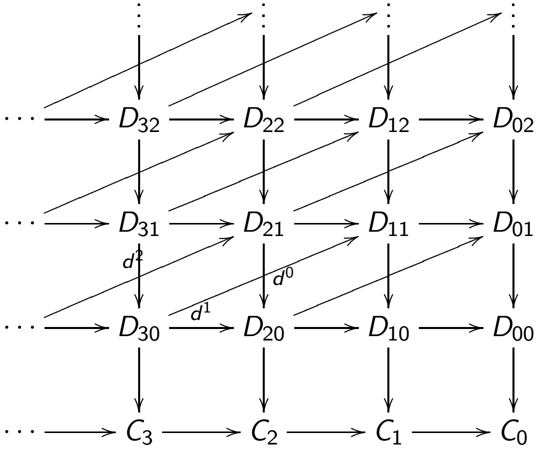
 & & \downarrow & & \downarrow{\scriptstyle d^0} & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & D_{30} & \xrightarrow{d^1} & D_{20} & \longrightarrow & D_{10} & \longrightarrow & D_{00} \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & C_3 & \longrightarrow & C_2 & \longrightarrow & C_1 & \longrightarrow & C_0
\end{array}
$$

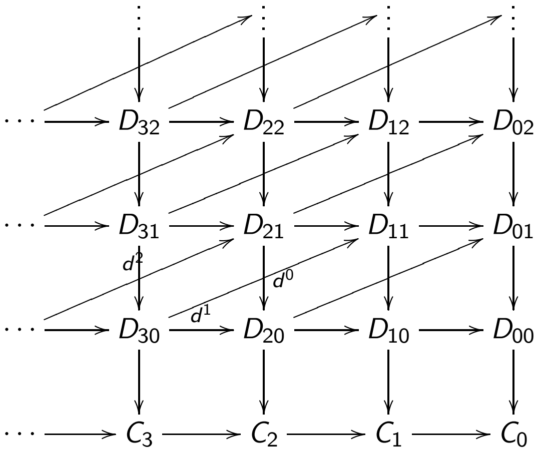
$$\partial = d^0 + d^1$$

but for $d^1 d^1 \neq 0$


D_{32} D_{22} D_{12} D_{02}
D_{31} D_{21} D_{11} D_{01}
D_{30} D_{20} D_{10} D_{00}
C_3 C_2 C_1 C_0
d^2
d^1
d^0



$$\partial = d^0 + d^1 + d^2$$



$$\partial = d^0 + d^1 + d^2$$

but for $d^2 d^2 \neq 0$ etc

Lemma (C.T.C. Wall)

There is a free A-resolution $R_* \to M$ with

$$R_n = \bigoplus_{p+q=n} D_{p,q}$$

and boundary homomorphism

$$\partial = d^0 + d^1 + d^2 + d^3 + \cdots$$

On any summand $D_{p,q}$ all but finitely many d^i are zero.

5. POTENTIAL COMPUTATIONS

SCENARIO 1.

Let N be a normal subgroup of G. Set $Q = G/N$.

Let C_* be a free $\mathbb{Z}Q$-resolution of $\mathbb{Z}$.

We can produce suitable resolutions D_{m*} from a free $\mathbb{Z}N$-resolution of $\mathbb{Z}$.

Wall's lemma was proved in this context and provides a free $\mathbb{Z}G$-resolution of $\mathbb{Z}$.

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Wall's lemma was proved in this context and provides a free $\mathbb{Z}G$-resolution of $\mathbb{Z}$.

This technique underlies HAP functions such as:

```
ResolutionNilpotentGroup(G,n)
ResolutionSubnormalSeries([G,N1,N2,...Nk],n)
```

Proposition

The free nilpotent group G of class two on 4 generators has integral cohomology groups

$$
\begin{array}{lll}
H^1(G,\mathbb{Z}) \cong \mathbb{Z}^4, & H^2(G,\mathbb{Z}) \cong \mathbb{Z}^{20}, & H^3(G,\mathbb{Z}) \cong \mathbb{Z}^{56}, \\
H^4(G,\mathbb{Z}) \cong \mathbb{Z}^{84}, & H^5(G,\mathbb{Z}) \cong \mathbb{Z}_3^4 \oplus \mathbb{Z}^{90}, & H^6(G,\mathbb{Z}) \cong \mathbb{Z}_3^4 \oplus \mathbb{Z}^{84}, \\
H^7(G,\mathbb{Z}) \cong \mathbb{Z}^{56}, & H^8(G,\mathbb{Z}) \cong \mathbb{Z}^{20}, & H^9(G,\mathbb{Z}) \cong \mathbb{Z}^4, \\
H^{10}(G,\mathbb{Z}) \cong \mathbb{Z}, & H^n(G,\mathbb{Z}) = 0 \ (n \geq 11). &
\end{array}
$$

The ring $H^*(G,\mathbb{Z})$ is generated by: 4 classes in degree 1, 20 classes in degree 2, 36 classes in degree 3 and 20 classes in degree 4.

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The ring $H^*(G,\mathbb{Z})$ is generated by: 4 classes in degree 1, 20 classes in degree 2, 36 classes in degree 3 and 20 classes in degree 4.

The additive structure of $H^*(G,\mathbb{Z})$ was first calculated in [Larry Lambe, "Cohomology of principal G-bundles over a torus when H*(BG,R) is polynomial", Bulletin Soc. Math. de Belgium, 38 (1986), 247-264].

```
gap> n:=3;;m:=7;;
gap> F:=FreeGroup(3);;G:=NilpotentQuotient(F,2);;
gap> R:=ResolutionNilpotentGroup(G,10);;
gap> for n in [1..m-1] do
> Print(``Cohomology in dimension '',n,`` = '',
> Cohomology(HomToIntegers(R),n),``\n''); od;
Cohomology in dimension 1 = [ 0, 0, 0 ]
Cohomology in dimension 2 = [ 0, 0, 0, 0, 0, 0, 0, 0 ]
Cohomology in dimension 3 = [ 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0 ]
Cohomology in dimension 4 = [ 0, 0, 0, 0, 0, 0, 0, 0 ]
Cohomology in dimension 5 = [ 0, 0, 0 ]
Cohomology in dimension 6 = [ 0 ]
gap> Dimension(R)(7);
0
gap>List([1..m-1],n->Length(IntegralRingGenerators(R,n)));
[ 3, 8, 6, 0, 0, 0]
```

```
gap> F:=FreeGroup(4);; G:=NilpotentQuotient(F,2);;

gap> LG:=LowerCentralSeriesLieAlgebra(G);;
gap> LieAlgebraHomology(LG,8);
[ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
  0, 0, 0, 0 ]
```

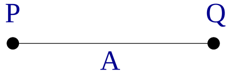
```
gap> F:=FreeGroup(4);; G:=NilpotentQuotient(F,2);;

gap> LG:=LowerCentralSeriesLieAlgebra(G);;
gap> LieAlgebraHomology(LG,8);
[ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
  0, 0, 0, 0 ]
```

Compare:

```
gap> GroupHomology(G,8);
[ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
  0, 0, 0, 0 ]
```

SCENARIO 2.

Let G be the fundamental group of a graph of groups. (For instance, an amalgamated free product $G := P *_A Q$ corresponding to the graph



).

Let C_* be the cellular chain complex of the graph.

We can produce suitable resolutions D_{m*} from free $\mathbb{Z}G_e$-resolutions for the edge groups G_e and free $\mathbb{Z}G_v$-resolutions for the vertex groups G_v.

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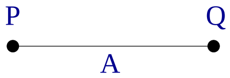

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Let C_* be the cellular chain complex of the graph.

We can produce suitable resolutions D_{m*} from free $\mathbb{Z}G_e$-resolutions for the edge groups G_e and free $\mathbb{Z}G_v$-resolutions for the vertex groups G_v.

This technique underlies the HAP function:
`ResolutionGraphOfGroups(G,n)`

The amalgamated free product $S_5 *_{S_3} S_4$ can be represented as a graph of groups.

```
gap> S5:=SymmetricGroup(5);
gap> S4:=SymmetricGroup(4);
gap> S3:=SymmetricGroup(3);

gap> S3S5:=GroupHomomorphismByFunction(S3,S5,x->x);;
gap> S3S4:=GroupHomomorphismByFunction(S3,S4,x->x);;
gap> D:=[S5,S4,[S3S5,S3S4]];;

gap> R:=ResolutionGraphOfGroups(D,8);;
gap> Homology(TensorWithIntegers(R),7);
[ 2, 2, 2, 4, 60 ]
```

So $H_7(S_5 *_{S_3} S_4, \mathbb{Z}) = (Z_2)^3 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_{60}$.

SCENARIO 3.

Let G act on some cellular contractible space X such that cells are permuted.

Let $C_* = C_*(X)$.

We can produce suitable resolutions D_{m*} from free $\mathbb{Z}G_e$-resolutions for the stabilizer groups $G_e \leq G$ of cells e.

Cellular space X can sometimes be produced using POLYMAKE software ...

Orbit Polytopes

Let $\alpha\colon G \to Gl(\mathbb{R}^n)$ be a faithful representation.
Let $v \in \mathbb{R}^n$ have trivial stabilizer group.

Definition
$P(G) = \text{Convex hull } \{\alpha g(v) : g \in G\}$

$G = A_4$ acts on $v = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ by

$$\alpha g(v) = (x_{g^{-1}(1)}, x_{g^{-1}(2)}, x_{g^{-1}(3)}, x_{g^{-1}(4)})$$

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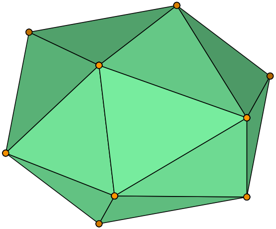
For $v = (1, 2, 3, 4)$ we get



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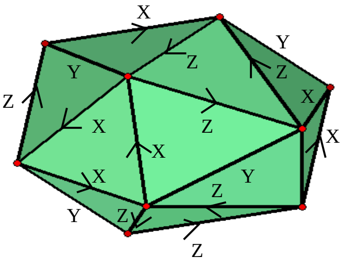

$x = (1, 3, 2), y = (1, 2)(3, 4), z = (2, 4, 3)$

The HAP function

`PolytopalComplex(G,v,n)`

uses POLYMAKE to produce n terms of the non-free resolution $C_*(X)$ and compute the stabilizer subgroups G_e.

Wall's lemma not yet implemented for this case.

The HAP function

`PolytopalComplex(G,v,n)`

uses POLYMAKE to produce n terms of the non-free resolution $C_*(X)$ and compute the stabilizer subgroups G_e.

Wall's lemma not yet implemented for this case.

But we can still use $C_*(X)$ to find a presentation of G.

```
gap> G:=SylowSubgroup(AlternatingGroup(18),3);;
gap> P:=PolytopalComplex(G,[1,2,3,4,5,6,7,8,9,
>10,11,12,13,14,15,16,17,18],2);
gap> PresentationOfResolution(P);
rec( freeGroup := <free group on the generators
 [ f1, f2, f3, f4, f5, f6, f7, f8 ]>,
 relators := [ f1^3, f2*f1*f2^-1*f1^-1, f3*f1*f3^-1*f1^-1,
 f4*f3*f4^-1*f1^-1, f1*f4^-1*f2^-1*f4, f5*f1*f5^-1*f1^-1,
 f6*f1*f6^-1*f1^-1, f7*f1*f7^-1*f1^-1, f8*f1*f8^-1*f1^-1,
 f2^3, f3*f2*f3^-1*f2^-1, f2*f4^-1*f3^-1*f4,
 f5*f2*f5^-1*f2^-1, f6*f2*f6^-1*f2^-1, f7*f2*f7^-1*f2^-1,
 f8*f2*f8^-1*f2^-1, f3^3, f5*f3*f5^-1*f3^-1,
 f6*f3*f6^-1*f3^-1, f7*f3*f7^-1*f3^-1, f8*f3*f8^-1*f3^-1,
 f4^3, f5*f4*f5^-1*f4^-1, f6*f4*f6^-1*f4^-1,
 f7*f4*f7^-1*f4^-1, f8*f4*f8^-1*f4^-1, f5^3,
 f6*f5*f6^-1*f5^-1, f7*f5*f7^-1*f5^-1, f8*f7*f8^-1*f5^-1,
 f5*f8^-1*f6^-1*f8, f6^3, f7*f6*f7^-1*f6^-1,
 f6*f8^-1*f7^-1*f8, f7^3, f8^3 ] )
```

POSSIBLE SCENARIO 4?

Resolutions in commutative algebra.